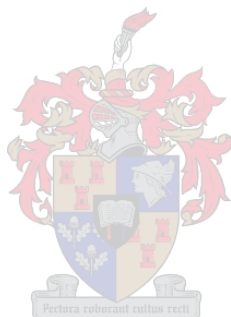


# INTERPOLATORY REFINABLE FUNCTIONS, SUBDIVISION AND WAVELETS

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# Declaration

I, the undersigned, hereby declare that this dissertation contains no material which has been accepted for a degree or diploma by the University of Stellenbosch or any other institution, except by way of background information and duly acknowledged in the dissertation, and that, to the best of my knowledge and belief, this dissertation contains no material previously published or written by another person, except where due acknowledgment is made in the text of the dissertation.

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# Summary

Subdivision is an important iterative technique for the efficient generation of curves and surfaces in geometric modelling. The convergence of a subdivision scheme is closely connected to the existence of a corresponding refinable function. In turn, such a refinable function can be used in the multi-resolutional construction method for wavelets, which are applied in many areas of signal analysis.

As an introduction to subdivision, we give in Chapter 1 a survey of some results in corner-cutting subdivision for curves, and then the following Chapters 2 to 6 are devoted to the topic of interpolatory subdivision for curves. First, in Chapter 2, after discussing Dubuc–Deslauriers subdivision, we introduce a general class  $\mathcal{A}_{\mu,\nu}$  of symmetric interpolatory subdivision schemes with the property of polynomial filling up to a given odd degree  $2\nu - 1$ . Also, we show in Theorem 2.7 that any member of  $\mathcal{A}_{\mu,\nu}$  is uniquely expressible in terms of a finite sequence of Dubuc–Deslauriers schemes.

In Chapter 3, we present two construction methods for convergent schemes in  $\mathcal{A}_{\mu,\nu}$ . The first method is based on the sampling at the half integers of a finitely supported function  $Q$  with appropriate properties, and the second method uses a Bezout identity containing a Hurwitz polynomial  $H$ .

We proceed to develop, in Section 4.2, and as ultimately stated in Theorem 4.8, a sufficient condition, consisting of two inequalities, for convergence, which provides an alternative to a well-known existing condition due to C.A. Micchelli, and which is then shown, in Section 4.3, to be applicable for certain subclasses of  $\mathcal{A}_{\mu,\nu}$ .

Next, in Chapter 5, we introduce, as an extension of Dubuc–Deslauriers subdivision, yet another construction method for schemes in  $\mathcal{A}_{\mu,\nu}$ , as based on the sampling at  $\frac{1}{2}$  of a certain fundamental interpolant sequence, and for which, as stated in Corollary 5.7, an efficient computational method is then derived by using the Dubuc–Deslauriers expansion result of Theorem 2.7. In the setting of splines, we then show that our Theorem 4.8 yields

convergence also for a subclass of subdivision schemes not satisfying the abovementioned Micchelli condition.

All of the above subdivision schemes are based on the availability of a bi-infinite initial data sequence. Since, in many practical applications, a given finite initial data sequence can not be extended in a natural way to be bi-infinite, we develop in Chapter 6, for the special case of Dubuc–Deslauriers subdivision, a modified subdivision scheme which is equivalent to Dubuc–Deslauriers subdivision away from the boundaries, and in such a way that the properties of interpolation and polynomial filling are preserved. Finally, in Chapter 7, we use the results of Chapter 6 to construct boundary-adapted interpolation wavelets, and then present applications in signature smoothing and image decomposition.

# Opsomming

Subdivisie (of onderverdeling) is 'n belangrike tegniek wat op 'n doeltreffende en vinnige manier krommes en oppervlakke genereer in geometriese modellering. Die konvergensie van 'n subdivisieskema is nou verwant aan die bestaan van 'n ooreenstemmende verfyndbare funksie. So 'n verfyndbare funksie kan in die multiresolusionele konstruksiemetode van golfies, wat toegepas word in baie gebiede van seinverwerking, gebruik word.

As inleiding tot subdivisie, gee ons in Hoofstuk 1 'n oorsig van sommige resultate van hoeksny subdivisie vir krommes. Hoofstukke 2 tot 6 word gewy aan die onderwerp van interpolerende subdivisie vir krommes. Eerstens, in Hoofstuk 2, na 'n bespreking van Dubuc–Deslauriers subdivisie, stel ons 'n algemene klas  $\mathcal{A}_{\mu,\nu}$  van simmetriese interpolerende subdivisieskemas, met die eienskap van polinoomvulling tot 'n gegewe onewe graad  $2\nu - 1$ , bekend. In Stelling 2.7 wys ons dan dat enige lid van  $\mathcal{A}_{\mu,\nu}$  op 'n unieke manier in terme van 'n eindige ry Dubuc–Deslauriers skemas uitdrukbaar is.

In Hoofstuk 3 gee ons twee konstruksiemetodes vir konvergente skemas in  $\mathcal{A}_{\mu,\nu}$ . Die eerste metode is gebaseer op die monsterring by die half-heelgetalle van 'n funksie  $Q$  met eindige steungebied en ander toepaslike eienskappe, terwyl die tweede metode gebruik maak van 'n Bezout identiteit wat 'n Hurwitz polinoom  $H$  bevat.

Ons gaan voort, in Afdeling 4.2, om 'n voldoende voorwaarde vir konvergensie, soos uiteindelik in Stelling 4.8 geformuleer, te ontwikkel. Hierdie voorwaarde, wat 'n alternatief tot die bekende voorwaarde deur C.A. Micchelli is, bestaan uit twee ongelykhede en word dan in Afdeling 4.3 op sekere subklasse van  $\mathcal{A}_{\mu,\nu}$  suksesvol toegepas.

Volgende, in Hoofstuk 5, stel ons, as 'n uitbreiding van Dubuc–Deslauriers subdivisie, 'n verdere konstruksiemetode vir skemas in  $\mathcal{A}_{\mu,\nu}$  voor. Hierdie metode is gebaseer op die monsterring by  $\frac{1}{2}$  van 'n sekere fundamentele interpolant, wat dan, soos geformuleer in Gevolg 5.7, lei tot 'n doeltreffende berekeningsmetode, waarin gebruik gemaak word van die Dubuc–Deslauriers uitbreidingsresultaat van Stelling 2.7. In die geval van latfunksies,

toon ons dan dat Stelling 4.8 konvergensie lewer vir 'n subklas van subdivisieskemas wat nie aan die bogenoemde Micchelli-voorwaarde voldoen nie.

Al bogenoemde subdivisieskemas is gebaseer op die beskikbaarheid van 'n dubbel-oneindige aanvanklike datary. Aangesien 'n gegewe eindige datary in baie praktiese toepassings nie op 'n natuurlike manier na 'n dubbel-oneindige ry uitgebrei kan word nie, ontwikkel ons in Hoofstuk 6, vir die spesiale geval van Dubuc–Deslauriers subdivisie, 'n rand-aangepaste subdivisieskema. Hierdie aangepaste subdivisieskema is ekwivalent aan Dubuc–Deslauriers subdivisie weg van die rante, en behou die interpolasie- en polinoomvullingseienskappe van Dubuc–Deslauriers naby die rante. Laastens, in Hoofstuk 7, gebruik ons die resultate van Hoofstuk 6 om rand-aangepaste interpolasiegolfies te kontrueer. Toepassings in handtekeningvergladding en in beelddekomposisie word aangebied.



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# List of Symbols

Symbol	Definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_k$	the set of natural numbers $\leq k$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_k$	the set of nonnegative integers $\leq k$
$\{j + \mathbb{N}_k\}$	the set of integers $\{j + 1, j + 2, \dots, j + k\}$
$\{j + \mathbb{Z}_k\}$	the set of integers $\{j, j + 1, \dots, j + k\}$
$\mathbb{J}_k$	the set of integers $\{-k + 1, \dots, k\}$
$\{j + \mathbb{J}_k\}$	the set of integers $\{j - k + 1, \dots, j + k\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^k$	the set $\{\mathbf{x} = (x_1, x_2, \dots, x_k) : x_j \in \mathbb{R}, j \in \mathbb{N}_k\}$
$\sum_j$	the sum $\sum_{j \in \mathbb{Z}}$
$\mathbb{C}$	the set of complex numbers
$\lfloor x \rfloor$	the largest integer $\leq x$
$\lceil x \rceil$	the smallest integer $\geq x$
$M(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences
$M_0(\mathbb{Z})$	the subspace of $M(\mathbb{Z})$ consisting of those sequences in $M(\mathbb{Z})$ with finite support, i.e. a finite number of non-zero elements
$M(\mathbb{R})$	the linear space of real-valued functions on $\mathbb{R}$
$M_0(\mathbb{R})$	the subspace of $M(\mathbb{R})$ consisting of finitely supported functions in $M(\mathbb{R})$
$\mathbf{a}$	(refinement) mask in $M_0(\mathbb{Z})$
$\text{supp}(\mathbf{a})$	support of the mask $\mathbf{a}$ , i.e. $\text{supp}(\mathbf{a}) = \{j : a_j \neq 0\}$
$A$	the mask symbol, defined by $\sum_j a_j(\cdot)^j$ (a Laurent polynomial or a polynomial) corresponding to the mask $\mathbf{a} \in M_0(\mathbb{Z})$
$S_{\mathbf{a}}$	subdivision operator with mask $\mathbf{a} \in M_0(\mathbb{Z})$
$S_{\mathbf{a}}^r$	subdivision operator, with mask $\mathbf{a}$ , applied $r$ times
$\mathbf{c}^{(r)}$	the resulting sequence after applying $S_{\mathbf{a}}^r$ to a sequence $\mathbf{c}$
$\Delta \mathbf{c}$	the backward difference sequence defined by $(\Delta \mathbf{c})_j = \Delta c_j = c_j - c_{j-1}$ , $j \in \mathbb{Z}$ , if $\mathbf{c} \in M(\mathbb{Z})$
$\Delta f$	the backward difference function defined by $f - f(\cdot - 1)$ , if $f \in M(\mathbb{R})$

Symbol	Definition
$\Delta^m f$	the $m$ -th backward difference function defined by $\Delta^m f = \Delta(\Delta^{m-1} f)$ , $m \geq 2$ , if $f \in M(\mathbb{R})$
$\gamma^{(r)}$	$c_j^{(r)} - \frac{1}{2} \left( c_{j-1}^{(r)} + c_{j+1}^{(r)} \right)$ , $j \in \mathbb{Z}$ , $r \in \mathbb{Z}_+$ , for $c^{(r)} \in M(\mathbb{Z})$
$\sup_j$	the supremum over all $j \in \mathbb{Z}$
$\sup_x$	the supremum over all $x \in \mathbb{R}$
$\ell^\infty(\mathbb{Z})$	the subspace of bounded sequences in $M(\mathbb{Z})$
$C(\mathbb{R})$	the linear space of continuous functions in $M(\mathbb{Z})$
$C_0(\mathbb{R})$	the subspace of $C(\mathbb{R})$ consisting of all finitely supported functions
$C^k(\mathbb{R})$	for $k \in \mathbb{Z}_+$ , $C^k(\mathbb{R}) := \{f \in M(\mathbb{R}) : f^{(j)} \in C(\mathbb{R}), j \in \mathbb{Z}_k\}$ , with the convention $f^{(0)} = f$
$C^{-1}(\mathbb{R})$	the subspace of $M(\mathbb{R})$ consisting of piecewise continuous functions
$C_u(\mathbb{R})$	the linear space of bounded functions in $C(\mathbb{R})$
$\ \cdot\ _\infty$	the sup norm for the linear space $\ell^\infty(\mathbb{Z})$ (or $C_u(\mathbb{R})$ )
$\Delta^\infty(\mathbb{Z})$	the subspace of $M(\mathbb{Z})$ consisting of those bi-infinite sequences $c \in M(\mathbb{Z})$ which are such that $\Delta c \in \ell^\infty(\mathbb{Z})$
$\Phi$	limit function of the subdivision scheme $S_a$
$\phi, \varphi$	refinable function with respect to a given mask
$\pi_n$	the linear space of polynomials of degree $\leq n$
$(\cdot)_+^k$	the truncated power, where $x_+^k = x^k$ if $x \geq 0$ , and $x_+^k = 0$ , if $x < 0$ , and with $0^0 = 1$
$S_m(\mathbb{Z})$	cardinal spline space of order $m$
$N_m$	cardinal B-spline of order $m$
$E_m$	the Euler–Frobenius polynomial $\sum_j N_m(j+1)(\cdot)^j$
$a^{(m)}$	$\left\{ \frac{1}{2^{m-1}} \binom{m}{j} : j \in \mathbb{Z} \right\}$ , the refinement mask of $N_m$
$A_m$	mask symbol associated with Lane–Riesenfeld subdivision $S_{a^{(m)}}$
$\Phi_m$	limit curve of Lane–Riesenfeld subdivision $S_{a^{(m)}}$
$\delta_j$	the Kronecker delta, equal to zero for all $j \in \mathbb{Z}$ , except for $\delta_0 = 1$
$\delta_{j,k}$	the Kronecker delta, equal to zero for all $j, k \in \mathbb{Z}$ , except for $\delta_{j,j} = 1$
$\delta$	the sequence $\{\delta_j : j \in \mathbb{Z}\}$
$\ell_{n,k}$	for $k \in \mathbb{J}_n$ , the Lagrange fundamental polynomials of degree $(2n-1)$ , with respect to the interpolation points $\mathbb{J}_n$

Symbol	Definition
$\mathbf{d}_n$	Dubuc–Deslauriers mask of order $n$ (satisfying the polynomial filling property for $p \in \pi_{2n-1}$ )
$\mathbf{D}_n$	Dubuc–Deslauriers mask symbol of order $n$
$\Phi_n^D$	limit function of the Dubuc–Deslauriers subdivision scheme $S_{\mathbf{d}_n}$
$\phi_n^D$	Dubuc–Deslauriers refinable function with refinement mask $\mathbf{d}_n$
$\mathcal{A}_{\mu,\nu}$	a class of symmetric interpolatory and polynomial filling mask symbols
$p^{(e)}$	the even part $\sum_j p_{2j}(\cdot)^{2j}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$
$p^{(o)}$	the odd part $\sum_j p_{2j+1}(\cdot)^{2j+1}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$
$\widehat{f}(x)$	for $x \in \mathbb{R}$ , the Fourier transform $\int_{-\infty}^{\infty} e^{-ixt} f(t) dt$ of $f \in M(\mathbb{R})$
$A(t \cdot)$	for $t \in \mathbb{R}$ , a linear combination of two consecutive Dubuc–Deslauriers mask symbols
$T_a$	cascade operator for a given mask $\mathbf{a} \in M_0(\mathbb{Z})$
$\alpha_j$	defined by $\sum_{k=j+1}^{n-1} (k-j)\mathbf{a}_{2k+1}$ , $j \in \mathbb{Z}_{n-2}$ , and zero otherwise, for $\mathbf{a} \in M_0(\mathbb{Z})$
$[x_0, \dots, x_n]f$	$n$ -th order divided difference of $f$ with respect to the points $x_0 < x_1 < \dots < x_n$
$\{\mathbf{R}_j : j \in \mathbb{J}_n\}$	a specific kind of fundamental interpolant with respect to the interpolation points $\mathbb{J}_n$
$\widetilde{N}_{2j}$	the B-spline with knots $\{-j+1, -j+2, \dots, 0, \frac{1}{2}, 1, \dots, j-1, j\}$
$\{\phi_j^r\}$	a sequence of refinable functions, constructed by adapting the Dubuc–Deslauriers refinable function $\phi_n^D$ to an interval
$M_r$	$\mathbb{R}^{2^r L+1}$
$\{S_r : r \in \mathbb{Z}_+\}$	the subdivision operator sequence for finite sequences in $M_r$
$\Phi_L$	the limit curve of the boundary-adapted Dubuc–Deslauriers subdivision scheme
$\mathbf{a}_{j,k}^{(r)}$	(refinement) mask of the boundary-adapted Dubuc–Deslauriers subdivision scheme
$\{\psi_j^r\}$	interpolation wavelets (constructed from boundary-adapted Dubuc–Deslauriers refinable functions $\{\phi_j^r\}$ )

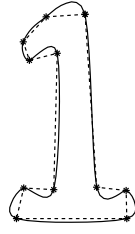




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# Introduction to subdivision

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Consider the following simple iterative procedure: For a given sequence  $\mathbf{c}^{(0)} = \{\mathbf{c}_j^{(0)} : j \in \mathbb{Z}\}$  in the plane, generate a new ‘denser’ sequence  $\mathbf{c}^{(1)} = \{\mathbf{c}_j^{(1)} : j \in \mathbb{Z}\}$  in the plane, where the odd-indexed elements of the new sequence interpolate the old ones. Alternatively one could demand that, with  $\Gamma^{(0)}$  and  $\Gamma^{(1)}$  denoting the polygons connecting the points of, respectively,  $\mathbf{c}^{(0)}$  and  $\mathbf{c}^{(1)}$ , that  $\Gamma^{(1)}$  “smoothes out”  $\Gamma^{(0)}$  in the sense of corner-cutting.

Subdivision schemes generate a new sequence by taking a linear combination of the old sequence, in contrast to standard interpolation (or smoothing) procedures which involve calculating an interpolatory (or smoothing) function and then evaluating the function. For example if one generates the new sequence using

$$\left. \begin{aligned} \mathbf{c}_{2j}^{(1)} &= \frac{1}{2}(\mathbf{c}_{j-1}^{(0)} + \mathbf{c}_j^{(0)}), \\ \mathbf{c}_{2j+1}^{(1)} &= \mathbf{c}_j^{(0)}, \end{aligned} \right\} j \in \mathbb{Z}, \quad (1.1)$$

then the even-indexed elements of the new sequence are generated halfway between the old ones. This step can of course be repeated indefinitely, roughly ‘doubling’ the number of points in the sequence at each step. In this case the new sequence fills in or converges to the polygon  $\Gamma^{(0)}$  connecting the initial sequence  $\mathbf{c}^{(0)}$  (see Figure 1.1). Thus we obtain, in the limit, a continuous piecewise linear curve. In general, the existence and smoothness of such a limit curve depend on the choice of the coefficients of the linear combination.

There is no unique or best way of obtaining the coefficients of the linear combination. In this chapter, we introduce the general concepts of subdivision schemes and then discuss choices for these coefficients that have a smoothing effect. A useful and general

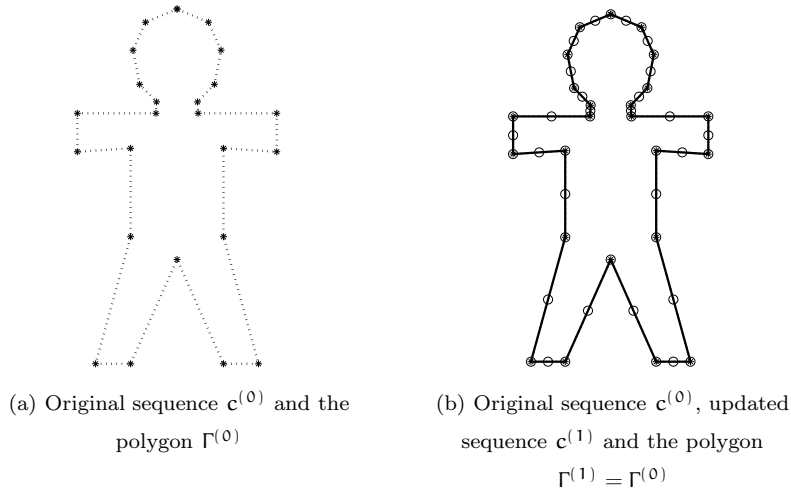


Figure 1.1: Illustration of iterative procedure (1.1)

introduction to subdivision methods can be found in [6].

## 1.1 Notation and general concepts

We shall denote by  $\mathbb{N}$  the set of natural numbers, by  $\mathbb{Z}$  the set of integers, by  $\mathbb{R}$  the set of real numbers and by  $\mathbb{C}$  the set of complex numbers. For the set of nonnegative integers we write  $\mathbb{Z}_+$  and for any  $k \in \mathbb{Z}_+$  we use the symbol  $\mathbb{Z}_k$  to denote the set of nonnegative integers  $\leq k$ , i.e.  $\mathbb{Z}_k := \{0, 1, \dots, k\}$  and the symbol  $\mathbb{N}_k$  to denote the set of positive integers  $\leq k$ , i.e.  $\mathbb{N}_k := \{1, 2, \dots, k\}$ .

We write  $M(\mathbb{Z})$  for the linear space of bi-infinite real-valued sequences, i.e.  $\mathbf{a} \in M(\mathbb{Z})$  if  $\mathbf{a} = \{a_j \in \mathbb{R} : j \in \mathbb{Z}\}$ , and use the notation  $\text{supp}(\mathbf{a}) = \{j : a_j \neq 0\}$  to denote the support of the sequence  $\mathbf{a}$ . The subspace of  $M(\mathbb{Z})$  consisting of those sequences in  $M(\mathbb{Z})$  with finite support will be denoted by  $M_0(\mathbb{Z})$ , i.e.  $\mathbf{a} = \{a_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$  if  $\mathbf{a} \in M(\mathbb{Z})$ , and  $\text{supp}(\mathbf{a})$  is a finite set.

Similarly, we write  $M(\mathbb{R})$  for the linear space of real-valued functions on  $\mathbb{R}$  and use the notation  $M_0(\mathbb{R})$  for the subspace of  $M(\mathbb{R})$  consisting of finitely supported functions in  $M(\mathbb{R})$ , i.e.  $f \in M_0(\mathbb{R})$  if  $f \in M(\mathbb{R})$ , and there exists a bounded interval  $[\alpha, \beta]$  such that  $f(x) = 0$ ,  $x \notin [\alpha, \beta]$ . The subspace of continuous functions in  $M_0(\mathbb{R})$  will be denoted

by  $C_0(\mathbb{R})$ .

For a given sequence  $\mathbf{a} \in M_0(\mathbb{Z})$ , we then generalise the subdivision algorithm implied by (1.1) by defining the **subdivision operator**  $S_{\mathbf{a}} : M(\mathbb{Z}) \rightarrow M(\mathbb{Z})$  by

$$(S_{\mathbf{a}}\mathbf{c})_j := \sum_k \mathbf{a}_{j-2k} \mathbf{c}_k, \quad j \in \mathbb{Z}, \quad \mathbf{c} \in M(\mathbb{Z}), \quad (1.2)$$

where we use the convention, here and throughout this thesis, that  $\sum_k = \sum_{k \in \mathbb{Z}}$ . The sequence  $\mathbf{a}$  is then called the **mask** of the subdivision operator  $S_{\mathbf{a}}$ , and the associated Laurent polynomial

$$A(z) := \sum_j \mathbf{a}_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \quad (1.3)$$

is called the **mask symbol** of the subdivision operator  $S_{\mathbf{a}}$ .

For any initial sequence  $\mathbf{c} \in M(\mathbb{Z})$ , the **subdivision scheme** associated with the mask  $\mathbf{a}$  generates the sequence  $\{\mathbf{c}^{(r)} : r \in \mathbb{Z}_+\}$  recursively by

$$\mathbf{c}^{(0)} = \mathbf{c}, \quad \mathbf{c}^{(r)} = S_{\mathbf{a}} \mathbf{c}^{(r-1)}, \quad r \in \mathbb{N}, \quad (1.4)$$

or, equivalently,

$$\mathbf{c}^{(0)} = \mathbf{c}, \quad \mathbf{c}^{(r)} = S_{\mathbf{a}}^r \mathbf{c}, \quad r \in \mathbb{N}. \quad (1.5)$$

Henceforth, for a given mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , whenever we refer to “the subdivision scheme  $S_{\mathbf{a}}$ ”, we shall mean the subdivision scheme (1.2), (1.4).

It should be noted that, whereas the definitions and results on subdivision throughout this thesis are stated and proved for initial sequences  $\mathbf{c} \in M(\mathbb{Z})$ , they can easily be extended, componentwise, to the case of vector-valued initial sequences  $\mathbf{c}$ . However, for simplicity of presentation, we restrict ourselves to the case where  $\mathbf{c} \in M(\mathbb{Z})$ , except possibly in the graphical examples, where we choose  $\mathbf{c} = \{\mathbf{c}_j : j \in \mathbb{Z}\}$ , with  $\mathbf{c}_j \in \mathbb{R}^2$ ,  $j \in \mathbb{Z}$ .

We denote by  $\ell^\infty(\mathbb{Z})$  the subspace of bounded sequences in  $M(\mathbb{Z})$ , i.e.  $\mathbf{c} \in \ell^\infty(\mathbb{Z})$  if  $\mathbf{c} \in M(\mathbb{Z})$ , and  $\|\mathbf{c}\|_\infty := \sup_j |\mathbf{c}_j| < \infty$ . Recall that  $\ell^\infty(\mathbb{Z})$  is a complete normed linear space with respect to the norm  $\|\cdot\|_\infty$ . For  $D \in \{M(\mathbb{Z}), M(\mathbb{R})\}$ , we define the backward

difference operator  $\Delta : D \rightarrow D$  by  $(\Delta c)_j = \Delta c_j = c_j - c_{j-1}$ ,  $j \in \mathbb{Z}$ , if  $D = M(\mathbb{Z})$  and by  $\Delta f = f - f(\cdot - 1)$  if  $D = M(\mathbb{R})$ . Also, we introduce the symbol  $\Delta^\infty(\mathbb{Z})$  to denote the subspace of  $M(\mathbb{Z})$  consisting of those bi-infinite sequences  $c \in M(\mathbb{Z})$  which are such that  $\Delta c \in \ell^\infty(\mathbb{Z})$ . Note in particular that  $\ell^\infty(\mathbb{Z})$  is a proper subspace of  $\Delta^\infty(\mathbb{Z})$ ; also,  $\Delta^\infty(\mathbb{Z})$  contains those unbounded sequences  $c \in M(\mathbb{Z})$  that are such that  $\Delta c \in \ell^\infty(\mathbb{Z})$ .

The following property of the subdivision operator  $S_a$  in (1.2) will be needed our subsequent work. We use, for  $x \in \mathbb{R}$ , the notation  $\lfloor x \rfloor$  for the largest integer  $\leq x$ , and  $\lceil x \rceil$  for the smallest integer  $\geq x$ .

**PROPOSITION 1.1** *For a given mask  $a \in M_0(\mathbb{Z})$ , the subdivision operator  $S_a$ , as defined by (1.2), satisfies*

$$S_a c \in \ell^\infty(\mathbb{Z}), \quad c \in \ell^\infty(\mathbb{Z}).$$

**Proof.** Suppose  $c \in \ell^\infty(\mathbb{Z})$ , and suppose  $\text{supp}(a) = \{M, M+1, \dots, N\}$  for some  $M, N \in \mathbb{Z}$ . Then, from (1.2), we have

$$\left| (S_a c)_{2j} \right| = \left| \sum_k a_{2j-2k} c_k \right| = \left| \sum_{k=\lceil M/2 \rceil}^{\lfloor N/2 \rfloor} a_{2k} c_{j-k} \right| \leq \|c\|_\infty \sum_{k=\lceil M/2 \rceil}^{\lfloor N/2 \rfloor} |a_{2k}|, \quad j \in \mathbb{Z}. \quad (1.6)$$

Similarly, we get

$$\left| (S_a c)_{2j+1} \right| \leq \|c\|_\infty \sum_{k=\lceil (M-1)/2 \rceil}^{\lfloor (N-1)/2 \rfloor} |a_{2k+1}|, \quad j \in \mathbb{Z}. \quad (1.7)$$

With the definition  $K = \max \left\{ \sum_{k=\lceil M/2 \rceil}^{\lfloor N/2 \rfloor} |a_{2k}|, \sum_{k=\lceil (M-1)/2 \rceil}^{\lfloor (N-1)/2 \rfloor} |a_{2k+1}| \right\}$ , it then follows from (1.6) and (1.7) that

$$\left| (S_a c)_j \right| \leq K \|c\|_\infty, \quad j \in \mathbb{Z}.$$

Hence,  $S_a c \in \ell^\infty(\mathbb{Z})$ . ■

We write  $C(\mathbb{R})$  for the linear space of continuous functions on  $\mathbb{R}$ , and, for  $k \in \mathbb{Z}_+$ , we define  $C^k(\mathbb{R}) := \{f \in M(\mathbb{R}) : f^{(j)} \in C(\mathbb{R}), j \in \mathbb{Z}_k\}$ , with the convention  $f^{(0)} = f$ . Observe

that  $C^0(\mathbb{R}) = C(\mathbb{R})$ . We shall use the symbol  $C^{-1}(\mathbb{R})$  to denote the space of piecewise continuous functions on  $\mathbb{R}$ .

The concept of convergence for a subdivision scheme is now defined as follows.

**DEFINITION 1.2** *For a given mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , we shall call the subdivision scheme  $S_{\mathbf{a}}$  **convergent on**  $M \subset M(\mathbb{Z})$  if, for every initial sequence  $\mathbf{c} \in M$ , there exists a function  $\Phi \in C(\mathbb{R})$  such that*

$$\left\| \Phi\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)} \right\|_{\infty} \longrightarrow 0, \quad r \longrightarrow \infty. \quad (1.8)$$

We call  $\Phi$  the **limit function** of the subdivision scheme  $S_{\mathbf{a}}$ .

Note in the definition above that, for any given  $x \in \mathbb{R}$ , since the dyadic set  $\{\frac{j}{2^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+\}$  is dense in  $\mathbb{R}$ , there exists a sequence  $\{j_r : r \in \mathbb{Z}_+\}$  such that  $\frac{j_r}{2^r} \longrightarrow x$ ,  $r \longrightarrow \infty$ , and thus

$$\left| \Phi(x) - \mathbf{c}_{j_r}^{(r)} \right| \leq \left| \Phi(x) - \Phi\left(\frac{j_r}{2^r}\right) \right| + \left| \Phi\left(\frac{j_r}{2^r}\right) - \mathbf{c}_{j_r}^{(r)} \right| \longrightarrow 0 + 0 = 0, \quad r \longrightarrow \infty,$$

from (1.8) and the fact that  $\Phi$  is continuous at  $x$ ; hence  $\mathbf{c}_{j_r}^{(r)} \longrightarrow \Phi(x)$ ,  $r \longrightarrow \infty$ .

Closely related to the convergence of subdivision schemes is the concept of a refinable function, as defined next.

**DEFINITION 1.3** *A function  $\phi \in C(\mathbb{R})$  is called **refinable** if there exists a sequence  $\mathbf{a} \in M_0(\mathbb{Z})$  such that*

$$\phi = \sum_j \mathbf{a}_j \phi(2 \cdot -j). \quad (1.9)$$

We call (1.9) the **refinement equation**, with corresponding **refinement mask**  $\mathbf{a}$ .

In the next section we discuss the cardinal B-splines as a family of refinable functions.

## 1.2 Cardinal B-splines and Lane–Riesenfeld subdivision

In this section we introduce, for  $m \in \mathbb{N}$ , the cardinal B-spline of order  $m$  as a finitely supported function, the integer-shifts of which provide a basis for the cardinal spline

space of order  $m$ . We then discuss the cardinal B-spline of order  $m$  as an example of a refinable function, with its associated subdivision scheme known in the literature as Lane–Riesenfeld subdivision [45].

For  $m \in \mathbb{N}$ , we define the *cardinal spline space*  $S_m(\mathbb{Z})$  as the set of all functions  $s \in M(\mathbb{R})$  which are such that

$$\left. \begin{aligned} s|_{[k, k+1)} &= p_k \in \pi_{m-1}, \quad k \in \mathbb{Z}, \\ s &\in C^{m-2}(\mathbb{R}), \end{aligned} \right\} \quad (1.10)$$

where, for a given  $k \in \mathbb{Z}_+$ , the symbol  $\pi_k$  denotes the space of polynomials of degree  $\leq k$ .

In order to find a basis for  $S_m(\mathbb{Z})$  consisting of the integer shifts of single finitely supported function, we define the cardinal B-splines  $N_m$  of order  $m \in \mathbb{N}$  recursively by

$$N_1(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & x \notin [0, 1), \end{cases} \quad (1.11)$$

$$N_m = \int_0^1 N_{m-1}(\cdot - t) dt, \quad m \geq 2. \quad (1.12)$$

With the truncated power function  $(\cdot)_+^k \in M(\mathbb{R})$  defined, for  $k \in \mathbb{Z}_+$ , by

$$x_+^k = \begin{cases} x^k, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

with the convention  $0^0 = 1$ , and the  $m$ -th backward difference function defined by  $\Delta^m f = \Delta(\Delta^{m-1} f)$ ,  $m \geq 2$ , for  $f \in M(\mathbb{R})$ , the cardinal B-spline  $N_m$  of order  $m \in \mathbb{N}$  satisfies (see e.g. [8, Chapter 4]) the following properties:

$$N_m = \frac{1}{(m-1)!} \Delta^m (\cdot)_+^{m-1} = \frac{1}{(m-1)!} \sum_{j \in \mathbb{Z}_m} (-1)^j \binom{m}{j} (\cdot - j)_+^{m-1}; \quad (1.13)$$

$$N_m(\cdot - j) \in S_m(\mathbb{Z}), \quad j \in \mathbb{Z}; \quad (1.14)$$

$$N_m = \sum_j a_j^{(m)} N_m(2 \cdot - j), \quad (1.15)$$

where the sequence  $\mathbf{a}^{(m)} = \{a_j^{(m)} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$  is defined by

$$a_j^{(m)} = \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}, \quad (1.16)$$

and where we have adopted the convention that  $\binom{m}{j} = 0$ ,  $j \notin \mathbb{Z}_m$ ;

$$N_m(x) = 0, \quad x \notin (0, m) \quad (\text{for } m \geq 2); \quad (1.17)$$

$$N_m(x) > 0, \quad x \in (0, m) \quad (\text{for } m \geq 2); \quad (1.18)$$

$$\sum_j N_m(x - j) = 1, \quad x \in \mathbb{R}; \quad (1.19)$$

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1), \quad x \in \mathbb{R}, \quad (\text{for } m \geq 2); \quad (1.20)$$

$$N'_m(x) = N_{m-1}(x) - N_{m-1}(x-1), \quad x \in \mathbb{R}, \quad (\text{for } m \geq 3); \quad (1.21)$$

$$N_m(x) = N_m(m-x), \quad x \in \mathbb{R}, \quad (\text{for } m \geq 2). \quad (1.22)$$

In particular, using (1.13), we obtain

$$N_2(x) = \begin{cases} x, & x \in (0, 1), \\ 2-x, & x \in [1, 2), \\ 0, & x \notin (0, 2). \end{cases} \quad (1.23)$$

Also, according to (1.15) and Definition 1.3, we see that the cardinal B-spline  $N_m$  is refinable with respect to the mask  $\mathbf{a} = \mathbf{a}^{(m)} \in M_0(\mathbb{Z})$  as given by (1.16). Examples for  $m = 2, 3, 4$  are plotted in Figure 1.2. The set  $\{N_m(\cdot - j) : j \in \mathbb{Z}\}$  is a basis for  $S_m(\mathbb{Z})$

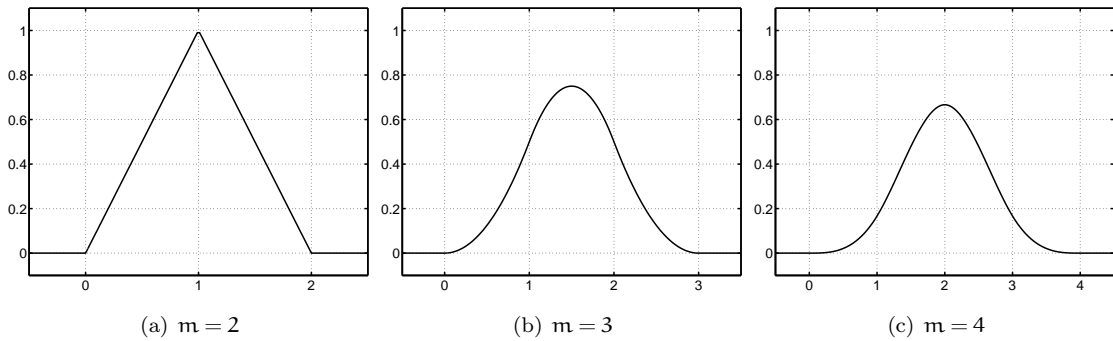


Figure 1.2: The cardinal B-splines  $N_m$  associated with refinement mask  $\mathbf{a}^{(m)}$ .

(see e.g. [50, Theorem 2.1]) in the sense that, for every  $s \in S_m(\mathbb{Z})$ , there exists a unique sequence  $\mathbf{c} \in M(\mathbb{Z})$  such that

$$s = \sum_j c_j N_m(\cdot - j).$$

Next, we consider the subdivision scheme  $S_{\mathbf{a}}$  associated with the mask  $\mathbf{a}^{(m)}$  in (1.16),

which is known as Lane–Riesenfeld subdivision (see e.g. [45] and [50]), and which, according to (1.2), (1.4) and (1.16), is given, for an initial sequence  $\mathbf{c} \in \mathbf{M}(\mathbb{Z})$ , by

$$\mathbf{c}^{(0)} = \mathbf{c}, \quad \mathbf{c}_j^{(r)} = \frac{1}{2^{m-1}} \sum_k \binom{m}{j-2k} \mathbf{c}_k^{(r-1)}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \quad (1.24)$$

The following convergence result for Lane–Riesenfeld subdivision was proved in [50, Theorem 2.2].

**THEOREM 1.4** *For any integer  $m \geq 2$ , the Lane–Riesenfeld subdivision scheme (1.24) converges on  $\Delta^\infty(\mathbb{Z})$ , with limit function*

$$\Phi = \Phi_m = \sum_j \mathbf{c}_j \mathbf{N}_m(\cdot - j). \quad (1.25)$$

Moreover, the convergence rate is geometric in the sense that, for every initial sequence  $\mathbf{c} \in \Delta^\infty(\mathbb{Z})$  in (1.24), we have

$$\|\Phi_m(\cdot/2^r) - \mathbf{c}^{(r)}\|_\infty \leq \frac{m-2}{2^r} \|\Delta \mathbf{c}\|_\infty, \quad r \in \mathbb{Z}_+. \quad (1.26)$$

Observe in particular from (1.25) in Theorem 1.4 that, if we choose the initial sequence  $\mathbf{c} = \delta = \{\delta_j : j \in \mathbb{Z}\}$  in (1.24), where

$$\delta_j := \begin{cases} 1, & j = 0, \\ 0, & j \neq 0, \end{cases} \quad j \in \mathbb{Z}, \quad (1.27)$$

we get  $\Phi_m = \mathbf{N}_m$ ; hence the Lane–Riesenfeld subdivision scheme (1.24) yields an efficient recursive algorithm for the computation of the cardinal B-splines. In fact, the graphs in Figure 1.2 were generated using this technique.

We proceed to consider Lane–Riesenfeld subdivision schemes for specific values of  $m$ . Setting  $m = 2$  in the Lane–Riesenfeld subdivision scheme (1.24), we obtain the recursive algorithm

$$\left. \begin{aligned} \mathbf{c}_{2j}^{(r)} &= (\mathbf{S}_a \mathbf{c}^{(r-1)})_{2j} = \frac{1}{2} (\mathbf{c}_{j-1}^{(r-1)} + \mathbf{c}_j^{(r-1)}), \\ \mathbf{c}_{2j+1}^{(r)} &= (\mathbf{S}_a \mathbf{c}^{(r-1)})_{2j+1} = \mathbf{c}_j^{(r-1)}, \end{aligned} \right\} \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \quad (1.28)$$



Note that (1.28) is identical to the algorithm implied by (1.1), where the odd-indexed elements of the updated sequence  $\mathbf{c}^{(r)}$  are simply the elements of the  $\mathbf{c}^{(r-1)}$ , while the even-indexed elements of the updated sequence  $\mathbf{c}^{(r)}$  are the midpoints between adjacent elements of the sequence  $\mathbf{c}^{(r-1)}$ , as illustrated in Figure 1.1.

Now observe from Theorem 1.4 that the limit curve for the subdivision scheme (1.28) is indeed the piecewise linear continuous function given by  $\Phi_2 = \sum_j \mathbf{c}_j \mathbf{N}_2(\cdot - j)$ , and that, from (1.28) and (1.26), we have  $\Phi_2\left(\frac{j}{2^r}\right) = \mathbf{c}_j^{(r)}$ ,  $j \in \mathbb{Z}$ ,  $r \in \mathbb{Z}_+$ , all of which is consistent with the example drawn in Figure 1.1. Observe also from (1.15) and (1.16) that  $\mathbf{N}_2$  is refinable with respect to the mask  $\mathbf{a} = \mathbf{a}^{(2)}$ .

Next, setting  $m = 3$  in the Lane–Riesenfeld subdivision scheme (1.24), we obtain the recursive algorithm

$$\left. \begin{aligned} \mathbf{c}_{2j}^{(r)} &= (\mathbf{S}_a \mathbf{c}^{(r-1)})_{2j} = \frac{1}{4} \left( \mathbf{c}_j^{(r-1)} + 3\mathbf{c}_{j-1}^{(r-1)} \right), \\ \mathbf{c}_{2j+1}^{(r)} &= (\mathbf{S}_a \mathbf{c}^{(r-1)})_{2j+1} = \frac{1}{4} \left( 3\mathbf{c}_j^{(r-1)} + \mathbf{c}_{j-1}^{(r-1)} \right), \end{aligned} \right\} j \in \mathbb{Z}, \quad r \in \mathbb{N}. \quad (1.29)$$

The algorithm (1.29) was originally described by de Rham [18] as a special case of a family of similar algorithms. He also proved that the limit curve for this special case is in  $C^1(\mathbb{R})$  and composed of quadratic arcs (see also [60]), whereas all the other algorithms in this family produce limit curves with fractal-like properties. Many years later, Chaikin analysed the algorithm (1.29) in [7], and it is therefore known as the de Rham–Chaikin algorithm.

According to Theorem 1.4, the subdivision scheme (1.29) converges to the (piecewise quadratic)  $C^1$ -smooth limit function  $\Phi_3 = \sum_j \mathbf{c}_j \mathbf{N}_3(\cdot - j)$  for any initial sequence  $\mathbf{c} \in \Delta^\infty(\mathbb{Z})$ . In Figure 1.3, a illustration is given for a specific choice of the initial sequence  $\mathbf{c}$ . Observe from (1.15) and (1.16) that  $\mathbf{N}_3$  is refinable with respect to the mask  $\mathbf{a} = \mathbf{a}^{(3)}$ .

In [60], Riesenfeld rediscovered that the de Rham–Chaikin algorithm (1.29) has a limit curve in  $C^1(\mathbb{R})$  and then in [45] Lane and Riesenfeld introduced the subdivision scheme

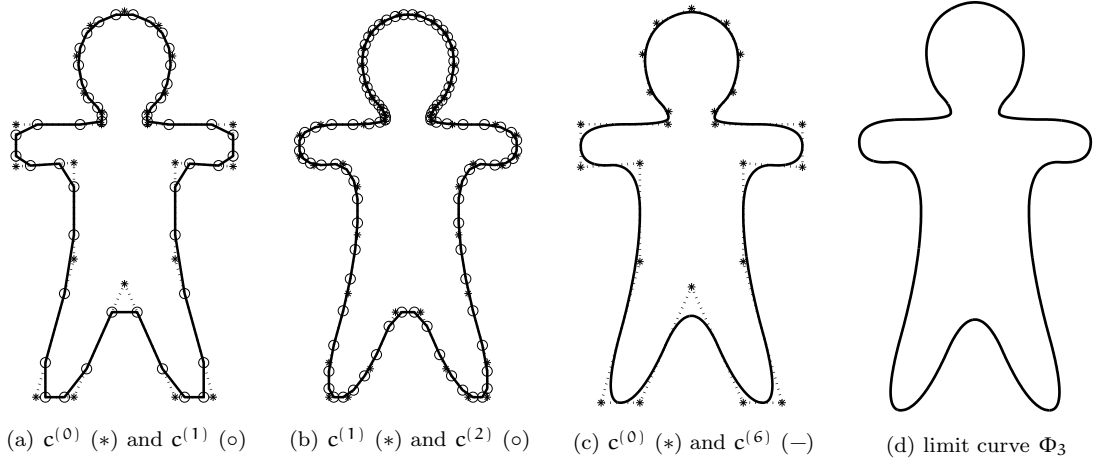


Figure 1.3: Subdivision with mask  $\mathbf{a}^{(3)}$ .

(1.24), thereby generalising the de Rham–Chaikin algorithm to include algorithms for generating cardinal splines of all orders  $\mathbf{m}$ . We show here the Lane–Riesenfeld subdivision scheme for  $\mathbf{m} = 4$  in (1.24), as given by

$$\left. \begin{aligned} c_{2j}^{(r)} &= \frac{1}{8}(c_j^{(r-1)} + 6c_{j-1}^{(r-1)} + c_{j-2}^{(r-1)}), \\ c_{2j+1}^{(r)} &= \frac{1}{8}(4c_j^{(r-1)} + 4c_{j-1}^{(r-1)}), \end{aligned} \right\} \quad j \in \mathbb{Z}, \quad r \in \mathbb{N},$$

(as illustrated in Figure 1.4); and for  $\mathbf{m} = 5$  in (1.24) we get

$$\left. \begin{aligned} c_{2j}^{(r)} &= \frac{1}{16}(c_j^{(r-1)} + 10c_{j-1}^{(r-1)} + 5c_{j-2}^{(r-1)}), \\ c_{2j+1}^{(r)} &= \frac{1}{16}(5c_j^{(r-1)} + 10c_{j-1}^{(r-1)} + c_{j-2}^{(r-1)}), \end{aligned} \right\} \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}.$$

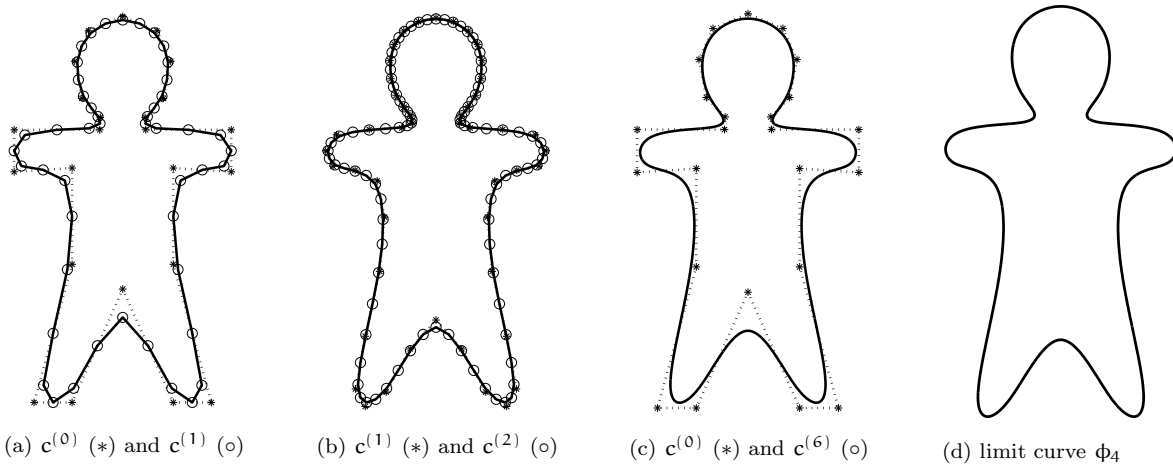


Figure 1.4: Subdivision with mask  $\mathbf{a}^{(4)}$ .

Since, for any  $m \in \mathbb{N}$ , we have the exact form (1.25) of the limit curve  $\Phi_m$  in terms translates of the cardinal B-spline of order  $m$ , we know the exact degree of smoothness of the limit curve, i.e.  $\Phi \in C^{m-2}(\mathbb{R})$ . Also, (1.11) and (1.20) can be used to derive explicit expressions for the associated refinable functions, the cardinal B-spline  $N_m$ . This is not the case in general: to our knowledge all other refinable functions are not known explicitly, and so need to be computed numerically by first solving an eigenvalue problem to find the values of the refinable function on  $\mathbb{Z}$ , before using the refinement equation (1.9) recursively to evaluate the refinable function on the dyadic set  $\{\frac{j}{2^r} : j \in \mathbb{Z}, r \in \mathbb{N}\}$  (see e.g. [55, Section 6.3]).

Note in particular for the Lane–Riesenfeld masks, from (1.16), (1.14) and the fact that  $S_m(\mathbb{Z}) \subset C^{m-2}(\mathbb{R})$ , that both the length of the mask and the regularity of the limit curve increases with  $m$ .

### 1.3 General positive masks

As an extension of Lane–Riesenfeld subdivision, we consider in this section subdivision schemes with positive masks  $\mathbf{a} \in M_0(\mathbb{Z})$  which, for a given  $n \in \mathbb{N}$ , satisfy the conditions

$$\text{supp}(\mathbf{a}) = \mathbb{Z}_n, \quad (1.30)$$

$$a_j > 0, \quad j \in \mathbb{Z}_n, \quad (1.31)$$

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \quad (1.32)$$

The following fundamental result, as proved in [50, Theorem 2.5 and Proposition 2.1] (see also [53] and [52]), extends the refinement result (1.15), (1.16) to a more general context.

**THEOREM 1.5** *For a given integer  $n \geq 2$ , suppose the sequence  $\mathbf{a} \in M_0(\mathbb{Z})$  satisfies the conditions (1.30), (1.31) and (1.32). Then there exists a refinable function  $\phi \in C_0(\mathbb{R})$*

with refinement mask  $\mathbf{a}$  such that

$$\phi(x) = 0, \quad x \notin (0, n), \quad (1.33)$$

$$\phi(x) > 0, \quad x \in (0, n), \quad (1.34)$$

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}. \quad (1.35)$$

Moreover,  $\phi$  is the unique function in  $C_0(\mathbb{R})$  satisfying (1.9) and (1.35).

Convergence of the subdivision scheme  $S_{\mathbf{a}}$  for masks satisfying the conditions of Theorem 1.5 was proved in [53] (see also [50, Theorem 2.5]) for initial sequences  $\mathbf{c} \in \ell^\infty(\mathbb{Z})$  in the subdivision scheme (1.4). Here we state a recent generalisation proved in [20, Theorem 3.1], that also allows for unbounded initial sequences  $\mathbf{c}$  in (1.4), as long as the corresponding difference sequence is bounded, i.e. if  $\Delta \mathbf{c} \in \ell^\infty(\mathbb{Z})$ , and which also provides an explicit bound (in terms of the mask) for the geometric convergence rate of the subdivision scheme  $S_{\mathbf{a}}$ .

**THEOREM 1.6** *Let the mask  $\mathbf{a} \in M_0(\mathbb{Z})$  satisfy the conditions of Theorem 1.5. Then the corresponding subdivision scheme  $S_{\mathbf{a}}$  is convergent on  $\Delta^\infty(\mathbb{Z})$ , and has the limit function  $\Phi \in C(\mathbb{R})$ , as given by*

$$\Phi = \sum_j c_j \phi(\cdot - j), \quad (1.36)$$

with  $\phi$  denoting the refinable function of Theorem 1.5. Moreover, the subdivision scheme  $S_{\mathbf{a}}$  converges geometrically in the sense that, with the number  $\rho = \rho(\mathbf{a})$  defined by

$$\rho := \frac{1}{2} \sup \left\{ \sum_\ell |a_{j-2\ell} - a_{k-2\ell}| : j, k \in \mathbb{Z}, |j - k| \leq n - 1 \right\}, \quad (1.37)$$

we have

$$\frac{1}{2} \leq \rho \leq 1 - \min\{a_0, a_1, \dots, a_n\} < 1, \quad (1.38)$$

and

$$\left\| \Phi\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)} \right\|_\infty \leq \rho^r (n - 1) \|\Delta \mathbf{c}\|_\infty, \quad r \in \mathbb{N}. \quad (1.39)$$

It is interesting to note from Theorem 1.4 that, for the Lane–Riesenfeld subdivision scheme (1.24), the analogue (1.26) of the estimate (1.39) actually holds with geometric constant  $\frac{1}{2}$ , which is consistent with the lower bound for  $\rho$  in (1.38).

As also noted after Theorem 1.4 in the context of Lane–Riesenfeld subdivision, we see that the choice  $c = \delta$ , in Theorem 1.6 yields  $\Phi = \phi$ ; hence the subdivision scheme  $S_a$  can be used as a recursive algorithm for the computation of the associated refinable function  $\phi$ .

However, whereas convergence of a subdivision scheme  $S_a$  implies the existence of a corresponding refinable function (see e.g. [50, Theorems 2.3 and 2.4], the converse is not necessarily true: the mere existence of a refinable function is, in general, not sufficient to ensure the convergence of the associated subdivision scheme (see e.g. [5, Proposition 2.3] [41], [54]). For example, as shown in [54, Example 2.1], the function  $\phi = N_2(\frac{\cdot}{3})$  is refinable with mask symbol  $A(z) = \frac{1}{2} + z^3 + \frac{1}{2}z^6$ ,  $z \in \mathbb{C}$ , illustrated in Figure 1.5, while the associated subdivision scheme is not convergent, since  $c_j^{(r)} = 0$ ,  $j \neq 0 \bmod 3$ , for all  $r \in \mathbb{N}$  (cf. Theorem 1.8 in the following section).

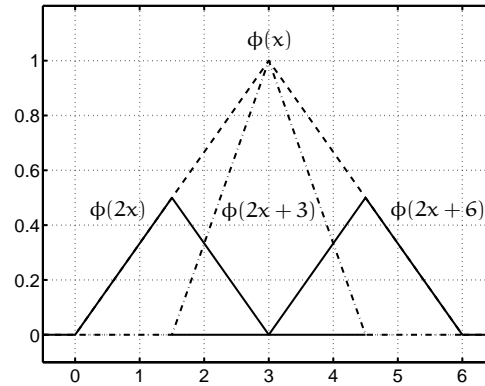


Figure 1.5: The refinable function  $\phi = N_2(\frac{\cdot}{3})$

Our next theorem, as proved in [50, Theorem 2.7], gives a result on the regularity (or minimum degree of smoothness) of the refinable function  $\phi$  in Theorem 1.5, and therefore also of the limit curve  $\Phi$ , as defined by (1.36) in Theorem 1.6.

Recall that a **Hurwitz polynomial** is defined as a polynomial with all its zeros in the open left half plane of  $\mathbb{C}$ . Hence the coefficients of a Hurwitz polynomial are necessarily

of the same sign.

**THEOREM 1.7** *In Theorem 1.5, suppose that, for  $n \geq 3$ , there exists an integer  $\nu \in \mathbb{N}$  and a Hurwitz polynomial  $C$  of degree  $\geq 1$ , such that the corresponding mask symbol  $A$ , as given by (1.3), satisfies*

$$A(z) = 2 \left( \frac{1+z}{2} \right)^{\nu+1} C(z), \quad z \in \mathbb{C}. \quad (1.40)$$

*Then  $\phi \in C^\nu(\mathbb{R})$ .*

Observe in particular that the conditions on the mask  $a$  in Theorem 1.5, together with (1.3), imply that  $\deg(A) = n$ , and thus, since  $\deg(C) \geq 1$ , we must have  $\nu \leq n - 2$ .

For example, for the choice  $C(z) = \frac{1}{4} + \frac{1}{4}z + \frac{1}{2}z^2$  in (1.40), we have from Theorem 1.7 that the regularity of associated refinable function  $\phi$  increases as the order of zero at  $z = -1$  increases, i.e. as  $\nu$  increases in (1.40). The resulting refinable functions plotted in Figure 1.6 support this fact.

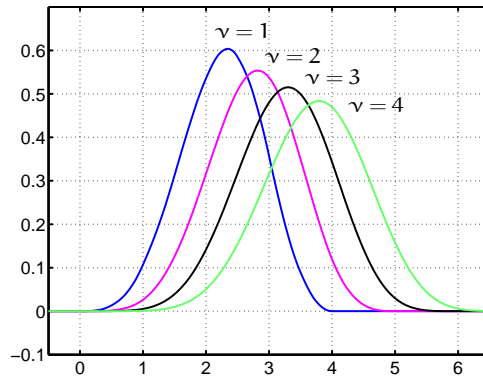


Figure 1.6: Refinable functions  $\phi$  with increasing smoothness

Observe from (1.16) and (1.3) that the mask symbol  $A_m$  corresponding to the Lane–Riesenfeld subdivision scheme is given by

$$A_m(z) = 2 \left( \frac{1+z}{2} \right)^m, \quad z \in \mathbb{C}. \quad (1.41)$$

Hence, if  $m \geq 3$ , the conditions of Theorem 1.7 are satisfied with  $\nu = m - 2$  and  $C(z) = \frac{1}{2}(1+z)$ ,  $z \in \mathbb{C}$ , so that, in this case, we have the smoothness result  $\phi = N_m \in C^{m-2}(\mathbb{R})$ , which is consistent with (1.14) and the bottom line of (1.10).

## 1.4 Nonnegative masks

The positivity condition (1.31) for the existence of a refinable function, as well as for subdivision convergence, can be weakened to include nonnegative masks, i.e. masks  $\mathbf{a} \in M_0(\mathbb{Z})$  that are such that  $\mathbf{a}_j \geq 0$ ,  $j \in \mathbb{Z}$ . It is known (see e.g. [5] or [64]) that for a nonnegative mask  $\mathbf{a} \in M_0(\mathbb{Z})$  and for  $n \geq 2$ , the conditions (1.32), together with the conditions

$$\text{supp}(\mathbf{a}) \subset \mathbb{Z}_n, \quad (1.42)$$

and

$$0 < \mathbf{a}_0, \mathbf{a}_n < 1 \quad \text{and} \quad \gcd\{j : \mathbf{a}_j \neq 0\} = 1, \quad (1.43)$$

are necessary for the convergence of the corresponding subdivision scheme  $S_{\mathbf{a}}$ .

The conjectured sufficiency of conditions (1.32), (1.42) and (1.43) for subdivision convergence is discussed in [5, p. 55] as an important open problem, and much work has been done since in attempts to prove it (see e.g. [35, 48, 43]). The following theorem, as proved by Wang in [65, Theorem 1.2], provides a result for a large subclass of such masks.

**THEOREM 1.8** *Suppose, for  $n \geq 2$ , a nonnegative mask  $\mathbf{a} \in M_0(\mathbb{Z})$  satisfies the conditions (1.32), (1.42), and  $0 < \mathbf{a}_0, \mathbf{a}_n < 1$ , and suppose that there exist integers  $r < p < q$  in  $\text{supp}(\mathbf{a})$  such that  $\gcd(q - r, p - r) = 1$  with  $q - r$  an even number. Then the associated subdivision scheme  $S_{\mathbf{a}}$  converges on  $\ell^\infty(\mathbb{Z})$  and there exists a corresponding refinable function  $\phi \in C_0(\mathbb{R})$  with refinement mask  $\mathbf{a}$ .*

For example, the mask symbol

$$A(z) = \frac{3}{20} + \frac{1}{8}z + \frac{3}{4}z^3 + \frac{7}{10}z^4 + \frac{1}{8}z^5 + \frac{3}{20}z^6, \quad z \in \mathbb{C}, \quad (1.44)$$

satisfies the conditions of Theorem 1.8, with  $n = 6$ ,  $r = 0$ ,  $p = 3$  and  $q = 4$ . Hence the associated subdivision scheme  $S_{\mathbf{a}}$  converges, as illustrated in Figure 1.7; moreover, there exists a corresponding continuous, finitely supported refinable function  $\phi$ , as illustrated by Figure 1.8.

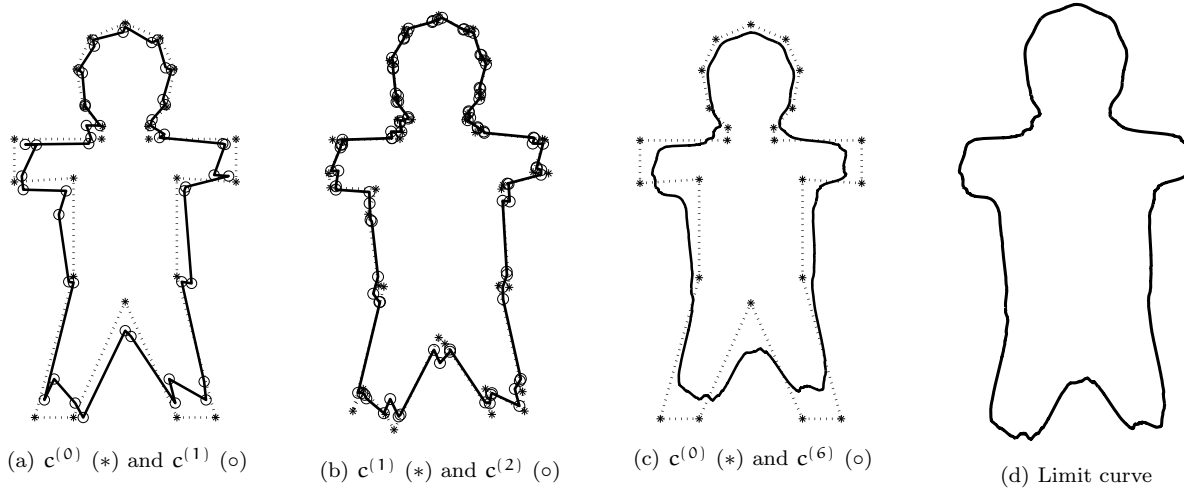


Figure 1.7: Subdivision with mask symbol  $A$  as given in (1.44)

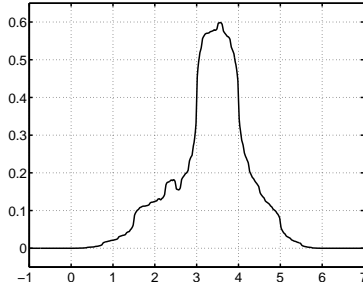


Figure 1.8: Refinable function  $\phi$  associated with mask symbol (1.44)

Subdivision schemes with positive masks are often referred to as ‘corner-cutting’ schemes, since this is a characteristic shared by all subdivision schemes with positive masks, as previously illustrated in Figures 1.3 and 1.4. However, in some applications it is important that the limit curve interpolates the initial data, suggesting the need for interpolatory subdivision schemes. A rather trivial example of such an interpolatory subdivision scheme is given by the Lane–Riesenfeld case  $m = 2$  in (1.28).

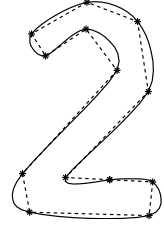
The subsequent Chapters 2 to 6 of this thesis are devoted to the construction and analysis of a general class of such interpolatory subdivision schemes.



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# The class $\mathcal{A}_{\mu,\nu}$ of symmetric interpolatory mask symbols

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In this chapter, after introducing the basic concepts of interpolatory subdivision, and discussing the special case of Dubuc–Deslauriers subdivision, we introduce and analyse a general class  $\mathcal{A}_{\mu,\nu}$  of symmetric, interpolatory subdivision schemes.

## 2.1 Preliminaries

We consider here the subclass of subdivision schemes  $S_a$  which are *interpolatory* in the sense that, in (1.2), we have

$$(S_a c)_{2j} = c_j, \quad j \in \mathbb{Z}, \quad c \in M(\mathbb{Z}). \quad (2.1)$$

If (2.1) holds, the sequence  $\{c^{(r)} : r \in \mathbb{Z}_+\}$  of real sequences generated by the subdivision scheme (1.4) satisfies

$$c_{2j}^{(r)} = c_j^{(r-1)}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}, \quad (2.2)$$

that is, at each step of the subdivision scheme the even-indexed elements of the updated sequence  $c^{(r)}$  correspond to the sequence  $c^{(r-1)}$ , whereas the odd-indexed elements of the updated sequence  $c^{(r)}$  are calculated as some weighted average of a finite number of neighbouring elements of  $c^{(r-1)}$ . This is, up to a single integer index shift, reminiscent of the Lane–Riesenfeld subdivision scheme (1.28) (with  $m = 2$ ), and which has, according to (1.41), the mask symbol

$$A_2(z) = \frac{1}{2} + z + \frac{1}{2}z^2, \quad z \in \mathbb{C}. \quad (2.3)$$

A necessary and sufficient condition on the mask  $\mathbf{a}$  for a subdivision scheme to be interpolatory is given in the following result.

**PROPOSITION 2.1** *For a mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , the subdivision scheme  $S_{\mathbf{a}}$  is interpolatory in the sense of (2.1) if and only if*

$$\mathbf{a}_{2j} = \delta_j, \quad j \in \mathbb{Z}, \quad (2.4)$$

*or, equivalently, if and only if the corresponding mask symbol  $A$ , as defined by (1.3), satisfies the identity*

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.5)$$

**Proof.** Suppose the sequence  $\mathbf{a} \in M_0(\mathbb{Z})$  satisfies (2.4). Then, from (1.2), we have for  $j \in \mathbb{Z}$  that

$$(S_{\mathbf{a}}\mathbf{c})_{2j} = \sum_{\mathbf{k}} \mathbf{a}_{2j-2\mathbf{k}} c_{\mathbf{k}} = \sum_{\mathbf{k}} \delta_{j-\mathbf{k}} c_{\mathbf{k}} = c_j,$$

thereby yielding (2.1).

If (2.1) holds for all  $\mathbf{c} \in M(\mathbb{Z})$ , we can choose  $\mathbf{c} = \delta$ , in (2.1) to deduce from (1.2) that

$$\delta_j = \sum_{\mathbf{k}} \mathbf{a}_{2j-2\mathbf{k}} \delta_{\mathbf{k}} = \mathbf{a}_{2j}, \quad j \in \mathbb{Z}$$

so that (2.4) holds.

It remains to prove the equivalence of (2.4) and (2.5). First use (1.3) to rewrite the left-hand side of (2.5), for  $z \in \mathbb{C} \setminus \{0\}$ , as

$$\begin{aligned} A(z) + A(-z) &= \sum_j \mathbf{a}_j z^j + \sum_j \mathbf{a}_j (-z)^j \\ &= \left[ \sum_j \mathbf{a}_{2j} z^{2j} + \sum_j \mathbf{a}_{2j+1} z^{2j+1} \right] + \left[ \sum_j \mathbf{a}_{2j} z^{2j} - \sum_j \mathbf{a}_{2j+1} z^{2j+1} \right], \end{aligned}$$

and thus

$$A(z) + A(-z) = 2 \sum_j \mathbf{a}_{2j} z^{2j}, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.6)$$

Now suppose that (2.4) holds. Then (2.6) gives

$$A(z) + A(-z) = 2 \sum_j \delta_j z^{2j} = 2, \quad z \in \mathbb{C} \setminus \{0\},$$

so that (2.5) holds. If (2.5) holds, then (2.6) implies

$$2 = A(z) + A(-z) = 2 \sum_j a_{2j} z^{2j}, \quad z \in \mathbb{C} \setminus \{0\},$$

and thus

$$\sum_j a_{2j} z^{2j} = 1, \quad z \in \mathbb{C} \setminus \{0\},$$

thereby yielding (2.4). ■

Observe that the Lane–Riesenfeld mask with  $m = 2$  and its associated mask symbol  $A_2$  as given in (2.3) is a shifted version of an interpolatory mask, in the sense that the mask symbol  $A(z) = z^{-1}A_2(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ , satisfies the interpolatory condition (2.5).

## 2.2 Dubuc–Deslauriers subdivision

In this section, as was done in [21], we derive Dubuc–Deslauriers subdivision as an optimally local polynomial filling subdivision scheme which is also interpolatory.

To this end, for a given  $n \in \mathbb{N}$ , consider the problem of finding a minimally supported mask  $\mathbf{a}$  such that the  $(2n - 1)$ -th degree polynomial filling property

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2n-1}, \quad (2.7)$$

holds.

For this purpose we introduce the Lagrange fundamental polynomials  $\ell_{n,k} \in \pi_{2n-1}$ , for  $k \in \mathbb{J}_n := \{-n + 1, \dots, n\}$ , as defined by

$$\ell_{n,k} = \prod_{k \neq j \in \mathbb{J}_n} \frac{\cdot - j}{k - j}, \quad k \in \mathbb{J}_n, \quad (2.8)$$

so that

$$\ell_{n,k}(j) = \delta_{k,j} := \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad k, j \in \mathbb{J}_n, \quad (2.9)$$

and

$$\sum_{k \in \mathbb{J}_n} p(k) \ell_{n,k} = p, \quad p \in \pi_{2n-1}. \quad (2.10)$$

Note, from (2.9) and (1.27), that  $\delta_{k,0} = \delta_k$ ,  $k \in \mathbb{Z}$ .

Setting  $j = 0$  and  $j = 1$  in (2.7), and then using (2.10) and (2.9), we obtain

$$\left. \begin{aligned} \mathbf{a}_{-2j} + \sum_{k \notin \mathbb{J}_n} \mathbf{a}_{-2k} \ell_{n,j}(k) &= \delta_j, \\ \mathbf{a}_{1-2j} + \sum_{k \notin \mathbb{J}_n} \mathbf{a}_{1-2k} \ell_{n,j}(k) &= \ell_{n,j}\left(\frac{1}{2}\right), \end{aligned} \right\} \quad j \in \mathbb{J}_n,$$

and therefore a necessary condition for a minimally supported mask  $\mathbf{a}$  to satisfy (2.7) is

$$\mathbf{a}_{-2j} = \delta_j, \quad j \in \mathbb{Z}, \quad (2.11a)$$

$$\mathbf{a}_{1-2j} = \ell_{n,j}\left(\frac{1}{2}\right), \quad j \in \mathbb{J}_n, \quad (2.11b)$$

$$\mathbf{a}_{1-2j} = 0, \quad j \notin \mathbb{J}_n. \quad (2.11c)$$

The choice (2.11) is also sufficient to fulfill (2.7). In fact, if  $j = 2m$ ,  $m \in \mathbb{Z}$ , then, for  $p \in \pi_{2n-1}$ , equation (2.11a) implies

$$\sum_k \mathbf{a}_{j-2k} p(k) = \sum_k \mathbf{a}_{2m-2k} p(k) = \sum_k \mathbf{a}_{2k} p(m-k) = p(m) = p\left(\frac{j}{2}\right),$$

whereas, if  $j = 2m + 1$ ,  $m \in \mathbb{Z}$ , then (2.11b),(2.11c) and (2.10) yield

$$\begin{aligned} \sum_k \mathbf{a}_{j-2k} p(k) &= \sum_k \mathbf{a}_{2k+1} p(m-k) \\ &= \sum_{k=-n}^{n-1} \ell_{n,-k}\left(\frac{1}{2}\right) p(m-k) \\ &= \sum_{k \in \mathbb{J}_n} p(m+k) \ell_{n,k}\left(\frac{1}{2}\right) = p\left(m + \frac{1}{2}\right) = p\left(\frac{j}{2}\right). \end{aligned}$$

Hence the subdivision scheme corresponding to the mask (2.11), as introduced by Dubuc and Deslauriers in [31, 28], is indeed a minimally supported mask sequence for which (2.7)

holds. We call, for a given  $n \in \mathbb{N}$ , the mask  $\mathbf{a} = \mathbf{d}_n = \{d_{n,j} : j \in \mathbb{Z}\}$  given by

$$\left. \begin{aligned} d_{n,2j} &= \delta_j, & j \in \mathbb{Z}, \\ d_{n,1-2j} &= \ell_{n,j} \left( \frac{1}{2} \right), & j \in \mathbb{J}_n, \\ d_{n,j} &= 0, & |j| \geq 2n, \end{aligned} \right\} \quad (2.12)$$

the **Dubuc–Deslauriers mask** of order  $n$ , and write

$$D_n(z) = \sum_j d_{n,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}, \quad (2.13)$$

for the associated **Dubuc–Deslauriers mask symbol**.

Observe that, since the two conditions (2.11a) and (2.4) are identical, we can conclude from Proposition 2.1 that the subdivision scheme with mask (2.11) is interpolatory. Accordingly, we call the interpolatory subdivision scheme  $S_{d_n}$  based on the choice  $\mathbf{a} = \mathbf{d}_n$ , the **Dubuc–Deslauriers subdivision scheme** of order  $n$ .

Note that, by construction, the mask  $\mathbf{a} = \mathbf{d}_n$  satisfies the polynomial filling property (2.7), i.e.

$$\sum_k d_{n,j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2n-1}, \quad (2.14)$$

and that the choice  $\mathbf{a} = \mathbf{d}_n$  is a mask of shortest possible length satisfying (2.7).

We now derive an explicit expression for the Dubuc–Deslauriers mask  $\mathbf{d}_n$ . To this end, we first calculate, for  $k \in \mathbb{J}_n$ ,

$$\begin{aligned} \prod_{k \neq j \in \mathbb{J}_n} \left( \frac{1}{2} - j \right) &= \prod_{k \neq j \in \mathbb{J}_n} \left( \frac{1-2j}{2} \right) \\ &= \frac{2}{1-2k} \prod_{j \in \mathbb{J}_n} \left( \frac{1-2j}{2} \right) \\ &= \frac{1}{2^{2n-1}} \frac{1}{1-2k} \prod_{j=-n}^{n-1} (2j+1) \\ &= \frac{(-1)^{n-1}}{2^{4n-3}} \frac{1}{2k-1} \left[ \frac{(2n-1)!}{(n-1)!} \right]^2, \end{aligned} \quad (2.15)$$

whereas

$$\prod_{k \neq j \in \mathbb{J}_n} (k-j) = (-1)^{n+k} (n-1+k)! (n-k)!,$$

from which, together with (2.12) and (2.8), we then deduce that the Dubuc–Deslauriers mask  $\mathbf{d}_n$  has the explicit formulation

$$\left. \begin{aligned} \mathbf{d}_{n,2j} &= \delta_j, & j \in \mathbb{Z}, \\ \mathbf{d}_{n,1-2j} &= \frac{n}{2^{4n-3}} \binom{2n-1}{n} \frac{(-1)^{j+1}}{2j-1} \binom{2n-1}{n-j}, & j \in \mathbb{J}_n, \\ \mathbf{d}_{n,j} &= 0, & |j| \geq 2n. \end{aligned} \right\} \quad (2.16)$$

For example, using (2.13) and (2.16), we obtain, for  $z \in \mathbb{C} \setminus \{0\}$ , the formulas

$$D_1(z) = \frac{1}{2} (z^{-1} + 2 + z), \quad (2.17)$$

$$D_2(z) = \frac{1}{16} (-z^{-3} + 9z^{-1} + 16 + 9z - z^3), \quad (2.18)$$

$$D_3(z) = \frac{1}{256} (3z^{-5} - 25z^{-3} + 150z^{-1} + 256 + 150z - 25z^3 + 3z^5). \quad (2.19)$$

Also, from (2.16), we observe that the mask coefficients  $\{\mathbf{d}_{n,j} : j \in \mathbb{Z}\}$  are symmetric, in the sense that

$$\mathbf{d}_{n,j} = \mathbf{d}_{n,-j}, \quad j \in \mathbb{Z}. \quad (2.20)$$

Next, in Section 2.3 below, we proceed to show that the Dubuc–Deslauriers subdivision scheme  $S_{\mathbf{d}_n}$  belongs to a general class of convergent interpolatory subdivision schemes.

## 2.3 An existence and convergence result

Following [51, Theorem 4.1 and Corollary 4.1], we next present a set of sufficient conditions on a mask symbol  $A$  for the existence of a corresponding interpolatory solution  $\phi \in C_0(\mathbb{R})$  of the refinement equation (1.9), and for the convergence of the associated interpolatory subdivision scheme  $S_a$ .

If a Laurent polynomial  $A$  satisfies  $A(z) = A(z^{-1})$ ,  $z \in \mathbb{C} \setminus \{0\}$ , we call  $A$  a *symmetric* Laurent polynomial. We then define the degree of a symmetric Laurent polynomial  $A(z) = \sum_j \alpha_j z^j$ ,  $z \in \mathbb{C} \setminus \{0\}$ , as the smallest integer  $m \in \mathbb{Z}_+$  for which it holds that  $\alpha_j = 0$ ,  $|j| \geq m+1$ .

**THEOREM 2.2** *For a given  $n \in \mathbb{N}$ , let the mask  $\mathbf{a} \in M_0(\mathbb{Z})$  be such that the mask symbol  $A$  defined by (1.3) has degree  $2n - 1$ , and satisfies the following properties:*

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}, \quad (2.21)$$

$$A(-1) = 0, \quad (2.22)$$

$$A(e^{ix}) > 0, \quad -\pi < x < \pi. \quad (2.23)$$

*Then there exists a refinable function  $\phi \in C(\mathbb{R})$ , with refinement mask  $\mathbf{a}$ , such that*

$$\phi(x) = 0, \quad x \notin (-2n + 1, 2n - 1); \quad (2.24)$$

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}; \quad (2.25)$$

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}. \quad (2.26)$$

*Moreover, the corresponding subdivision scheme  $S_{\mathbf{a}}$ , as given by (1.2) and (1.4), is interpolatory in the sense of (2.1), and converges on  $M(\mathbb{Z})$ , where the limit function  $\Phi \in C(\mathbb{R})$  is given by*

$$\Phi = \sum_j c_j \phi(\cdot - j), \quad (2.27)$$

*and where*

$$c_j^{(r)} = \Phi\left(\frac{j}{2^r}\right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (2.28)$$

**Remark:** Laurent polynomials which are real-valued on the unit circle in  $\mathbb{C}$  can be shown to be necessarily symmetric. Hence our positivity condition (2.23) above implies that Theorem 2.2 admits only symmetric Laurent polynomials, in which case the degree is well defined.

**Proof.** For the proof of the existence of a refinable function  $\phi \in C_0(\mathbb{R})$  satisfying the properties (2.24) – (2.26), we refer to [51, Theorem 4.1] or [38, Theorem 4.2].

Observing from (2.21) and Proposition 2.1 that the subdivision scheme  $S_{\mathbf{a}}$  is interpolatory in the sense of (2.1), it therefore remains to prove that the subdivision scheme  $S_{\mathbf{a}}$  is

convergent on  $M(\mathbb{Z})$ . To this end, it will suffice to prove that the function  $\Phi \in C(\mathbb{R})$  satisfies (2.28) for every initial sequence  $\mathbf{c} \in M(\mathbb{Z})$  in (1.4), since (1.8) then holds trivially for every  $\mathbf{c} \in M(\mathbb{Z})$ .

If  $r = 0$ , then (2.28) follows from (2.25) and (2.27). For  $r \geq 1$ , we use the refinability (1.9) of  $\phi$ , together with (1.2), (1.4) and (2.25) to deduce, for  $k \in \mathbb{Z}$ , that

$$\begin{aligned}
\Phi\left(\frac{j}{2^r}\right) &= \sum_k c_k \phi\left(\frac{j}{2^r} - k\right) \\
&= \sum_k c_k \left[ \sum_\ell a_\ell \phi\left(\frac{j}{2^{r-1}} - 2k - \ell\right) \right] \\
&= \sum_k c_k \sum_\ell a_{\ell-2k} \phi\left(\frac{j}{2^{r-1}} - \ell\right) \\
&= \sum_\ell (S_a \mathbf{c})_\ell \phi\left(\frac{j}{2^{r-1}} - \ell\right) \\
&= \sum_\ell c_\ell^{(1)} \phi\left(\frac{j}{2^{r-1}} - \ell\right) = \cdots = \sum_\ell c_\ell^{(r)} \phi(j - \ell) = c_j^{(r)},
\end{aligned}$$

thereby proving (2.28). ■

Observe that (2.21) and (2.22) imply

$$A(1) = 2. \tag{2.29}$$

We call a refinable function  $\phi \in C(\mathbb{R})$  an ***interpolatory refinable function*** if it also satisfies the interpolatory condition (2.25).

**Remarks:**

- (a) As opposed the “corner-cutting” subdivision schemes of Chapter 1, convergent interpolatory subdivision schemes have, according to (2.28), the property that for each  $r \in \mathbb{Z}_+$ , the sequence  $\mathbf{c}^{(r)}$  lies entirely on the limit curve and therefore “fills up” the limit curve  $\Phi$  as  $r$  increases.
- (b) In general the existence of a refinable function does not guarantee the convergence of the associated subdivision scheme, see e.g. [5, 41, 54]. Here however, for a mask which is such that its symbol  $A$  satisfies (2.21), the proof of the convergence of the



subdivision scheme  $S_a$  is based solely on the existence of an interpolatory refinable function  $\phi$ . We can conclude that for interpolatory subdivision schemes, the existence of an interpolatory refinable function is a necessary and sufficient condition for the convergence of the associated subdivision scheme.

The above observation is consistent with the result [5, Proposition 2.3] that if a function  $\phi \in C_0(\mathbb{R})$  is  $L^\infty$  stable, in the sense that there exists a constant  $A > 0$ , such that

$$A \|c\|_\infty \leq \left\| \sum_j c_j \phi(\cdot - j) \right\|_\infty, \quad c \in \ell^\infty(\mathbb{Z}), \quad (2.30)$$

and refinable with refinement mask  $a$ , then the associated subdivision scheme  $S_a$  is convergent. The fact that our interpolatory refinable functions above indeed satisfy the stability result (2.30) follows from the result [42, Theorem 5.1] according to which (2.30) is equivalent to the linear independence condition

$$\sum_j c_j \phi(\cdot - j) = 0 \text{ implies } c_j = 0, \quad j \in \mathbb{Z}, \quad c \in \ell^\infty(\mathbb{Z}),$$

which in our case is directly deducible from the interpolatory condition (2.25).

Following the argument introduced in [51], our next result shows that Theorem 2.2 can be used to prove the convergence of the Dubuc–Deslauriers subdivision scheme.

**THEOREM 2.3** *For  $n \in \mathbb{N}$ , the Dubuc–Deslauriers subdivision scheme  $S_{d_n}$ , with mask  $d_n$  as in (2.12), is convergent on  $M(\mathbb{Z})$ , and the limit function  $\Phi = \Phi_n^D$  is given by*

$$\Phi_n^D = \sum_j c_j \phi_n^D(\cdot - j).$$

*Moreover, the Dubuc–Deslauriers refinable function  $\phi_n^D \in C_0(\mathbb{R})$  satisfies the properties*

$$\begin{aligned} \phi_n^D &= \sum_j d_{n,j} \phi_n^D(2 \cdot - j); \\ \phi_n^D(x) &= 0, \quad x \notin (-2n + 1, 2n - 1); \end{aligned} \quad (2.31)$$

$$\phi_n^D(j) = \delta_j, \quad j \in \mathbb{Z}; \quad (2.32)$$

$$\sum_j \phi_n^D(x - j) = 1, \quad x \in \mathbb{R}.$$

**Proof.** According to Theorem 2.2, it will suffice to prove that the choice  $\mathbf{A} = \mathbf{D}_n$  satisfies the properties (2.21), (2.22) and (2.23).

The top line of (2.12) and Proposition 2.1 show that (2.21) holds. Next, choosing  $p(x) = 1$ ,  $x \in \mathbb{R}$  in (2.14), we find that

$$\sum_k d_{n,j-2k} = 1, \quad j \in \mathbb{Z}, \quad (2.33)$$

and thus the choice  $\mathbf{a} = \mathbf{d}_n$  satisfies the sum conditions (1.32), so that also (2.22) holds.

Finally, we use formulas proved in [51, Lemma 3.1], according to which

$$D_n(e^{ix}) = 2 \frac{\int_{-\pi}^x (\sin \omega)^{2n-1} d\omega}{\int_{-\pi}^{\pi} (\sin \omega)^{2n-1} d\omega}, \quad x \in \mathbb{R}, \quad (2.34)$$

and thus

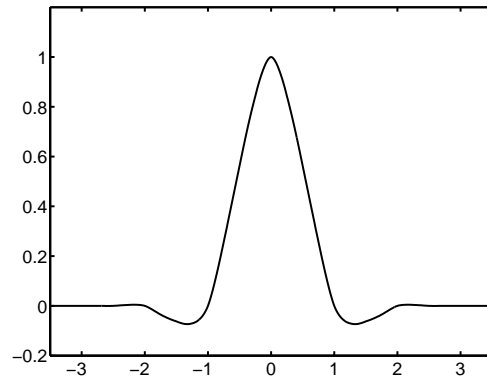
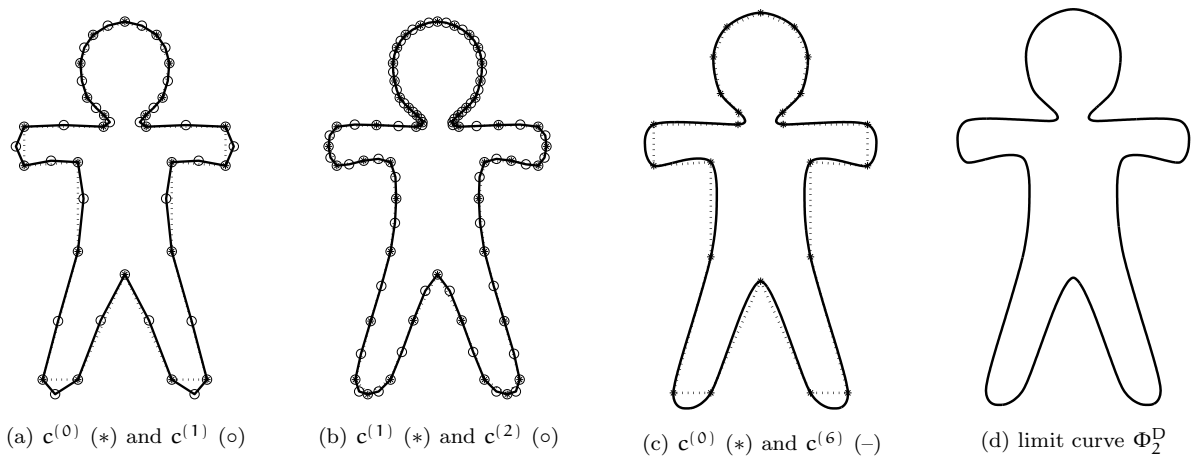
$$D_n(e^{ix}) = \frac{(2n-1)!}{2^{2n-2}[(n-1)!]^2} \int_x^\pi (\sin \omega)^{2n-1} d\omega > 0, \quad -\pi < x < \pi, \quad (2.35)$$

to deduce that (2.23) is also satisfied. ■

In [28, Theorem 6.2] the authors used, in contrast to (2.35), a Rolle-type argument to conclude that the choice  $\mathbf{A} = \mathbf{D}_n$ , as given by (2.12) and (2.13), satisfies the condition (2.23) for any  $n \in \mathbb{N}$ .

We call the refinable function  $\phi_n^D$  the ***Dubuc–Deslauriers refinable function***. For the analysis of the regularity (or smoothness) class of the Dubuc–Deslauriers refinable functions, which is outside the scope of this thesis, we refer to [28, Chapter 7; see in particular Theorem 7.11].

For example, with the mask  $\mathbf{a} = \mathbf{d}_2$ , as implied by (2.13) and (2.18), the corresponding refinable function  $\phi_2^D$  is plotted, using Dubuc–Deslauriers subdivision with initial sequence  $\mathbf{c} = \delta$ , in Figure 2.1. Also, the convergence of Dubuc–Deslauriers subdivision with  $n = 2$  is illustrated in Figure 2.2.


 Figure 2.1: Dubuc–Deslauriers refinable function  $\phi_2^D$ 

 Figure 2.2: Illustration of the Dubuc–Deslauriers subdivision scheme  $\mathcal{S}_{d_2}$ .

## 2.4 The general class $\mathcal{A}_{\mu,\nu}$

The Dubuc–Deslauriers mask  $\mathbf{d}_n$  satisfies the  $(2n-1)$ -th degree polynomial filling property (2.14). Also, since (2.20) holds, the corresponding mask symbol  $D_n$  is a symmetric Laurent polynomial:  $D_n(z) = D_n(z^{-1})$ ,  $z \in \mathbb{C} \setminus \{0\}$ ; and, as follows from the top line of (2.16) and Proposition 2.1, we also have  $D_n(z) + D_n(-z) = 2$ ,  $z \in \mathbb{C} \setminus \{0\}$ , i.e., the mask symbol  $D_n$  determines an interpolatory subdivision scheme. These observations motivate our next definition, which was first introduced in [22].

**DEFINITION 2.4** *For  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ , we say that a Laurent polynomial  $A$ , as given by (1.3), and with degree at most  $2(\mu + \nu) - 1$ , determines a symmetric interpolatory*

subdivision scheme  $S_a$  of accuracy  $2\nu - 1$  if it satisfies

$$A(z) = A(z^{-1}), \quad z \in \mathbb{C} \setminus \{0\}, \quad (2.36)$$

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}, \quad (2.37)$$

and

$$\sum_k a_{j-2k} p(k) = p(\tfrac{j}{2}), \quad j \in \mathbb{Z}, \quad p \in \pi_{2\nu-1}. \quad (2.38)$$

We denote the class of all such Laurent polynomials by  $\mathcal{A}_{\mu,\nu}$ .

Observe in particular that

$$\mathcal{A}_{0,\nu} = \{D_\nu\}, \quad \nu \in \mathbb{N}, \quad (2.39)$$

which follows from the fact, as established in Section 2.2, that, by construction, the Dubuc–Deslauriers mask symbol  $D_n$  is the unique symmetric Laurent polynomial of lowest possible degree such that the associated mask  $d_n$  satisfies the polynomial filling property (2.38), with  $\nu = n$ .

Also note that, by choosing the polynomial  $p$  in (2.38) as  $p(x) = 1$ ,  $x \in \mathbb{R}$ , we obtain

$$\sum_k a_{j-2k} = 1, \quad j \in \mathbb{Z}, \quad (2.40)$$

which, in turn, holds if and only if the sum conditions (1.32) hold. Therefore, if  $A \in \mathcal{A}_{\mu,\nu}$ , then

$$A(1) = 2, \quad A(-1) = 0. \quad (2.41)$$

Hence every mask symbol  $A \in \mathcal{A}_{\mu,\nu}$  has a zero at  $-1$ , and, moreover, satisfies the sum conditions (1.32).

Our next result shows that the polynomial reproduction property (2.38) can be replaced by an equivalent condition on the order of the zero at  $-1$  of the symbol  $A$ . For  $j \in \mathbb{Z}_+$ , we use the notation  $A^{(j)}$  to denote the  $j$ th derivative of the Laurent polynomial  $A$ , where  $A^{(0)} = A$ .

PROPOSITION 2.5 *For a mask  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$ , suppose  $\nu \in \mathbb{N}$ , and suppose the Laurent polynomial  $A$  defined by (1.3) is such that the interpolatory condition (2.37) holds. Then the mask  $\mathbf{a}$  satisfies the polynomial filling property (2.38) if and only if*

$$A^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_{2\nu-1}. \quad (2.42)$$

**Proof.** Since (2.37) and Proposition 2.1 give  $\mathbf{a}_{2j} = \delta_j$ ,  $j \in \mathbb{Z}$ , we see that the condition (2.38) is equivalent to the condition

$$\sum_k \mathbf{a}_{2k+1} p(j-k) = p(j + \tfrac{1}{2}), \quad j \in \mathbb{Z}, \quad p \in \pi_{2\nu-1}. \quad (2.43)$$

It therefore remains to prove that (2.43) holds if and only if (2.42) holds.

Since  $\mathbf{a}_{2j} = \delta_j$ ,  $j \in \mathbb{Z}$ , we see from (1.3) that  $A(z) = 1 + \sum_k \mathbf{a}_{2k+1} z^{2k+1}$ ,  $z \in \mathbb{C} \setminus \{0\}$ , and thus

$$A^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k q_j(2k+1) \mathbf{a}_{2k+1}, \quad j \in \mathbb{Z}_{2\nu-1}, \quad (2.44)$$

where, for  $x \in \mathbb{R}$ ,

$$q_0(x) = 1, \quad q_j(x) = \prod_{\ell \in \mathbb{Z}_{j-1}} (x - \ell), \quad j \in \mathbb{N}_{2\nu-1}. \quad (2.45)$$

Observe that  $q_j \in \pi_j$ ,  $j \in \mathbb{Z}_{2\nu-1}$ . Hence, if we define

$$p_j = q_j(-2 \cdot + 1 + 2j), \quad j \in \mathbb{Z}_{2\nu-1}, \quad (2.46)$$

then also  $p_j \in \pi_j$ ,  $j \in \mathbb{Z}_{2\nu-1}$ . Moreover,

$$A^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k p_j(j-k) \mathbf{a}_{2k+1}, \quad j \in \mathbb{Z}_{2\nu-1}, \quad (2.47)$$

and

$$p_j(j + \tfrac{1}{2}) = q_j(0) = \delta_j, \quad j \in \mathbb{Z}_{2\nu-1}, \quad (2.48)$$

from (2.45).

Suppose (2.43) holds. Then, since  $p_j \in \pi_j \subset \pi_{2\nu-1}$ ,  $j \in \mathbb{Z}_{2\nu-1}$ , we find from (2.47) and (2.48) that, for any integer  $j \in \mathbb{Z}_{2\nu-1}$ ,

$$\begin{aligned} A^{(j)}(-1) &= \delta_j + (-1)^{j+1} \sum_k p_j(j-k) a_{2k+1} \\ &= \delta_j + (-1)^{j+1} p_j(j + \tfrac{1}{2}) = \delta_j[1 + (-1)^{j+1}] = 0, \end{aligned}$$

thereby proving that (2.42) holds.

Conversely, suppose (2.42) holds. Then, from (2.44), we have

$$\sum_k q_j(2k+1) a_{2k+1} = (-1)^j \delta_j, \quad j \in \mathbb{Z}_{2\nu-1}. \quad (2.49)$$

Suppose now  $p \in \pi_{2\nu-1}$ , and fix  $j \in \mathbb{Z}$ . From (2.45), we see that  $\{q_\ell : \ell \in \mathbb{Z}_{2\nu-1}\}$  is a basis for  $\pi_{2\nu-1}$ . Hence, from (2.46), we deduce that  $\{p_\ell(\cdot - j + \ell) : \ell \in \mathbb{Z}_{2\nu-1}\}$  is a basis for  $\pi_{2\nu-1}$ . Thus there exists a coefficient sequence  $\{\alpha_{j,\ell} : \ell \in \mathbb{Z}_{2\nu-1}\} \subset \mathbb{R}$  such that  $p = \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} p_\ell(\cdot - j + \ell)$ . Using (2.46) and (2.49), we obtain

$$\begin{aligned} \sum_k a_{2k+1} p(j-k) &= \sum_k a_{2k+1} \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} p_\ell((j-k) - j + \ell) \\ &= \sum_k a_{2k+1} \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} p_\ell(-k + \ell) \\ &= \sum_k a_{2k+1} \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} q_\ell(-2(-k + \ell) + 1 + 2\ell) \\ &= \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} \sum_k a_{2k+1} q_\ell(2k+1) \\ &= \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} (-1)^\ell \delta_\ell = \alpha_{j,0}. \end{aligned} \quad (2.50)$$

But, from (2.48), we have that

$$\begin{aligned} p(j + \tfrac{1}{2}) &= \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} p_\ell(j + \tfrac{1}{2} - j + \ell) \\ &= \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} p_\ell(\ell + \tfrac{1}{2}) = \sum_{\ell \in \mathbb{Z}_{2\nu-1}} \alpha_{j,\ell} \delta_\ell = \alpha_{j,0}. \end{aligned} \quad (2.51)$$

It follows from (2.50) and (2.51) that (2.43) indeed holds. ■

Note that from (1.3), the symmetry condition (2.36) has the equivalent formulation

$$\mathbf{a}_j = \mathbf{a}_{-j}, \quad j \in \mathbb{Z}, \quad (2.52)$$

and the condition  $\deg(\mathbf{A}) \leq 2(\mu + \nu) - 1$  is equivalent to the condition

$$\mathbf{a}_j = 0, \quad j \notin \{-2(\mu + \nu) + 1, \dots, 2(\mu + \nu) - 1\}. \quad (2.53)$$

It follows, using also Proposition 2.5, that the class  $\mathcal{A}_{\mu,\nu}$  can, equivalently to Definition 2.4, be defined by the properties (2.53), (2.52), (2.37) and (2.42).

Generalising the Dubuc–Deslauriers result of [21, Theorem 2.1], we now establish further properties of the refinable function associated with a mask symbol in  $\mathcal{A}_{\mu,\nu}$ .

**THEOREM 2.6** *For  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ , suppose  $\mathbf{A} \in \mathcal{A}_{\mu,\nu}$  is such that there exists an associated refinable function  $\phi \in C_0(\mathbb{R})$ , and such that  $\phi$  is interpolatory in the sense of (2.25). Then  $\phi$  also satisfies the properties*

$$\sum_j p(j) \phi(\cdot - j) = p, \quad p \in \pi_{2\nu-1}; \quad (2.54)$$

$$\phi = \phi(-\cdot); \quad (2.55)$$

$$\phi\left(\frac{j}{2}\right) = \mathbf{a}_j, \quad j \in \mathbb{Z}. \quad (2.56)$$

If, moreover, for  $\mathbf{n} = \mu + \nu$ , we have the finite support property (2.24), then also

$$\phi\left(2\mathbf{n} - 1 - 2^{-m}\left(\mathbf{n} - \frac{3}{2} - k\right)\right) = 0, \quad m, k \in \mathbb{Z}_+. \quad (2.57)$$

**Proof.** To prove (2.54), suppose  $\ell \in \mathbb{Z}_{2\nu-1}$ ,  $k \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ . We shall prove that

$$\sum_j j^\ell \phi\left(\frac{k}{2^r} - j\right) = \left(\frac{k}{2^r}\right)^\ell, \quad (2.58)$$

which then implies (2.54), since the set  $\{\frac{k}{2^r} : k \in \mathbb{Z}, r \in \mathbb{Z}_+\}$  is dense in  $\mathbb{R}$ , and  $\phi$  is a finitely supported continuous function on  $\mathbb{R}$ .

Noting that (2.58) is an immediate consequence of (2.25) if  $r = 0$ , we assume next that  $r \geq 1$ . Then, using consecutively the refinability (1.9) of  $\phi$ , the polynomial filling property

(2.38) and the interpolatory property (2.25) of  $\phi$ , we get

$$\begin{aligned}
 \sum_j j^\ell \phi\left(\frac{k}{2^r} - j\right) &= \sum_j j^\ell \sum_m a_m \phi\left(\frac{k}{2^{r-1}} - 2j - m\right) \\
 &= \sum_m \left[ \sum_j a_{m-2j} j^\ell \right] \phi\left(\frac{k}{2^{r-1}} - m\right) \\
 &= \frac{1}{2^r} \sum_m m^\ell \phi\left(\frac{k}{2^{r-1}} - m\right) = \cdots = \left(\frac{1}{2^r}\right)^r \sum_m m^\ell \phi(k - m) = \left(\frac{k}{2^r}\right)^\ell,
 \end{aligned}$$

thereby completing the proof of (2.58).

Similarly, the symmetry (2.55) of  $\phi$  will be proved if we can show that, for  $k \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\phi\left(\frac{k}{2^r}\right) = \phi\left(-\frac{k}{2^r}\right). \quad (2.59)$$

For  $r = 0$ , (2.59) follows from the interpolatory property (2.25) of  $\phi$ , whereas for  $r = 1$ , it follows from the refinability (1.9) of  $\phi$ , the symmetry (2.52) of the mask  $a$ , and (2.25), that, for  $k \in \mathbb{Z}$ ,

$$\phi\left(-\frac{k}{2}\right) = \sum_j a_j \phi(-k-j) = \sum_j a_{-j} \phi(-k+j) = \sum_j a_j \phi(-k+j) = \sum_j a_j \phi(k-j) = \phi\left(\frac{k}{2}\right).$$

For  $r \geq 2$ , we use the refinability of  $\phi$ , together with (1.2), (2.52) and (2.25), to deduce that

$$\begin{aligned}
 \phi\left(-\frac{k}{2^r}\right) &= \sum_j a_j \phi\left(-\frac{k}{2^{r-1}} - j\right) \\
 &= \sum_j a_{-j} \phi\left(-\frac{k}{2^{r-1}} + j\right) \\
 &= \sum_j a_j \phi\left(-\frac{k}{2^{r-1}} + j\right) \\
 &= \sum_j a_j \left[ \sum_\ell a_\ell \phi\left(-\frac{k}{2^{r-2}} + 2j - \ell\right) \right] \\
 &= \sum_j a_j \left[ \sum_\ell a_{-\ell} \phi\left(-\frac{k}{2^{r-2}} + 2j + \ell\right) \right] \\
 &= \sum_j a_j \left[ \sum_\ell a_\ell \phi\left(-\frac{k}{2^{r-2}} + 2j + \ell\right) \right]
 \end{aligned}$$



$$\begin{aligned}
 &= \sum_{\ell} \left[ \sum_j a_{\ell-2j} a_j \right] \phi \left( -\frac{k}{2^{r-2}} + \ell \right) \\
 &= \sum_{\ell} (S_a a)_{\ell} \phi \left( -\frac{k}{2^{r-2}} + \ell \right) \\
 &\quad \vdots \\
 &= \sum_{\ell} (S_a^{r-1} a)_{\ell} \phi(-k + \ell) = (S_a^{r-1} a)_k,
 \end{aligned} \tag{2.60}$$

A similar argument shows that also  $\phi\left(\frac{k}{2^r}\right) = (S_a^{r-1} a)_k$ ,  $k \in \mathbb{Z}$ , which, together with (2.60), proves (2.59) for  $r \geq 2$ .

Property (2.56) is an immediate consequence of the refinability (1.9) and the interpolatory property (2.25) of  $\phi$ .

Finally, as was done in [21, Theorem 2.1], we prove (2.57) by induction. For  $m = 0$ , property (2.57) follows from (2.56) and the fact that  $\text{supp}(a) \subset [-2n+1, 2n-1]$ , since  $\deg(A) \leq 2n-1$ . To advance the inductive hypothesis from  $m$  to  $m+1$ , we use the refinability of  $\phi$  and the finite support property of  $a$  to deduce that, for  $k \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 \phi\left(2n-1-2^{-(m+1)}\left(n-\frac{3}{2}-k\right)\right) &= \sum_j a_j \phi\left(4n-2-2^{-m}\left(n-\frac{3}{2}-k\right)-j\right) \\
 &= \sum_{j=0}^{4n-2} a_{2n-1-j} \phi\left(2n-1-2^{-m}\left(n-\frac{3}{2}-[k+2^m j]\right)\right),
 \end{aligned}$$

thereby completing our inductive proof. ■

### Remarks:

(a) The refinability of  $\phi$  and equation (2.56) imply

$$\phi = \sum_j \phi\left(\frac{j}{2}\right) \phi(2 \cdot -j). \tag{2.61}$$

(b) The symmetry (2.55) of  $\phi$  and (2.57) imply that

$$\phi\left(-2n+1+2^{-m}\left(n-\frac{3}{2}-k\right)\right) = 0, \quad m, k \in \mathbb{Z}_+. \tag{2.62}$$

- (c) For  $k \geq n-1$ , (2.57) and (2.62) hold because the argument of  $\phi$  falls outside its support interval  $[-2n+1, 2n-1]$ , while for  $k \in \mathbb{Z}_{n-2}$ , the argument falls within the support. This, in turn, implies that, for  $n = \mu + \nu \geq 2$ , the refinable function  $\phi$  has an infinite number of zeros within its support and that these zeros are clustered more densely towards the edges of the support, as illustrated for the Dubuc–Deslauriers refinable function  $\phi_3^D$  with the associated mask symbol  $D_3 \in \mathcal{A}_{0,3}$  in Figure 2.3.

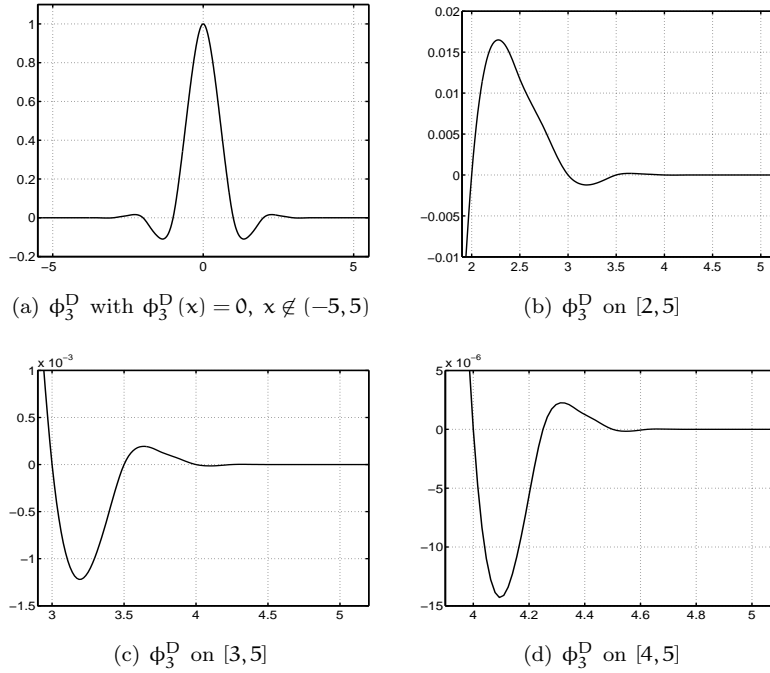


Figure 2.3: The clustered zeros of the refinable function  $\phi_3^D$

- (d) Since for  $n \in \mathbb{N}$ , the Dubuc–Deslauriers mask symbol  $D_n$  is in  $\mathcal{A}_{0,n}$ , we have from Theorems 2.6 and 2.3 that the corresponding refinable function  $\phi_n^D$  satisfies the polynomial reproduction property (2.54), i.e.

$$\sum_j p(j) \phi_n^D(\cdot - j) = p, \quad p \in \pi_{2n-1}, \quad (2.63)$$

and is symmetric, i.e.

$$\phi_n^D(-\cdot) = \phi_n^D. \quad (2.64)$$

- (e) Apart from the regularity result [5, Proposition 2.5 and Remark 2.6] (see also [51, Theorem 2.2]), not much is known about the regularity (smoothness) class of the

refinable function  $\phi$  of Theorem 2.6. It remains an interesting open problem to investigate this issue.

## 2.5 The Dubuc–Deslauriers expansion of $A \in \mathcal{A}_{\mu,\nu}$

Our result below, as was first established in [22], gives us a convenient representation for the class  $\mathcal{A}_{\mu,\nu}$  of mask symbols in terms of Dubuc–Deslauriers mask symbols.

**THEOREM 2.7** *For  $\mu \in \mathbb{Z}_+$ ,  $\nu \in \mathbb{N}$ , a Laurent polynomial  $A$  belongs to the class  $\mathcal{A}_{\mu,\nu}$  if and only if there exists a unique sequence  $\{t_j, j \in \mathbb{Z}_\mu\} \subset \mathbb{R}$ , with  $\sum_{j \in \mathbb{Z}_\mu} t_j = 1$ , such that*

$$A = \sum_{j \in \mathbb{Z}_\mu} t_j D_{\nu+j}, \quad (2.65)$$

and with  $\{D_n : n \in \mathbb{N}\}$  denoting the Dubuc–Deslauriers mask symbols as given by (2.12) and (2.13).

**Proof.** Suppose  $A \in \mathcal{A}_{\mu,\nu}$ . The symmetry (2.36) and interpolatory property (2.37) of the symbol  $A$  allow us to uniquely associate the Laurent polynomial  $A$  with the vector  $(a_1, a_3, \dots, a_{2(\nu+\mu)-1}) \in \mathbb{R}^{\nu+\mu}$ . For a fixed  $\ell \in \mathbb{Z}_{\nu-1}$ , we now choose  $p = p_\ell = (-2 \cdot +1)^{2\ell}$  in (2.38) to deduce that

$$\sum_k a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} = \sum_k a_{1-2k} \left(-k + \frac{1}{2}\right)^{2\ell} = \frac{1}{2^{2\ell}} \sum_k a_{1-2k} p_\ell(k) = \frac{1}{2^{2\ell}} p_\ell\left(\frac{1}{2}\right) = \delta_\ell. \quad (2.66)$$

But, from (2.52), we also have

$$\begin{aligned} \sum_k a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} &= \sum_{k \in \mathbb{Z}_{\mu+\nu-1}} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} + \sum_{k=-\mu-\nu}^{-1} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} \\ &= \sum_{k \in \mathbb{Z}_{\mu+\nu-1}} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} + \sum_{k=-\mu-\nu}^{-1} a_{-2k-1} \left(k + \frac{1}{2}\right)^{2\ell} \\ &= \sum_{k \in \mathbb{Z}_{\mu+\nu-1}} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} + \sum_{k \in \mathbb{Z}_{\mu+\nu-1}} a_{2k+1} \left(-k - \frac{1}{2}\right)^{2\ell} \\ &= 2 \sum_{k \in \mathbb{Z}_{\mu+\nu-1}} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell}. \end{aligned} \quad (2.67)$$

Combining (2.66) and (2.67), we find that the conditions

$$\sum_{k \in \mathbb{Z}_{\nu+\mu-1}} a_{2k+1} \left(k + \frac{1}{2}\right)^{2\ell} = \frac{1}{2} \delta_\ell, \quad \ell \in \mathbb{Z}_{\nu-1}, \quad (2.68)$$

are satisfied by the vector  $(a_1, a_3, \dots, a_{2(\nu+\mu)-1}) \in \mathbb{R}^{\nu+\mu}$ .

We proceed to rewrite the homogeneous system of linear equations for  $\ell \in \mathbb{N}_{\nu-1}$  in (2.68) as a matrix equation, and then use the concepts of rank and null space to prove our theorem. To this end, we define  $x_k = \left(k + \frac{1}{2}\right)^2$ ,  $k \in \mathbb{Z}_{\nu+\mu-1}$ , and observe that the points  $x_k$  are distinct, with  $x_k \neq 0$ ,  $k \in \mathbb{Z}_{\nu+\mu-1}$ . Hence the  $(\nu-1) \times (\nu+\mu)$  matrix  $X = (x_{\ell, k})_{\ell=1, \dots, \nu-1}^{k=0, \dots, \nu+\mu-1}$ , where  $x_{\ell, k} = (x_k)^\ell$ , is, according to a standard result for Vandermonde matrices with distinct points, of rank  $(\nu-1)$ . It follows that the null space  $\mathcal{N}(X)$  of  $X$  has dimension  $(\mu+1)$ .

According to the  $(\nu-1)$  homogeneous equations in (2.68), i.e. for  $\ell \in \mathbb{N}_{\nu-1}$ , we deduce that the vector  $(a_1, a_3, \dots, a_{2(\nu+\mu)-1}) \in \mathbb{R}^{\nu+\mu}$  is an element of  $\mathcal{N}(X)$ . Moreover, since  $D_{\nu+j} \in \mathcal{A}_{\mu, \nu}$ ,  $j \in \mathbb{Z}_\mu$ , we see that the vectors  $(d_{\nu+j, 1}, d_{\nu+j, 3}, \dots, d_{\nu+j, 2(\nu+j)-1}, 0, \dots, 0) \in \mathbb{R}^{\nu+\mu}$ ,  $j \in \mathbb{Z}_\mu$ , all belong to the nullspace  $\mathcal{N}(X)$ . Now note that the vectors  $(d_{\nu+j, 1}, d_{\nu+j, 3}, \dots, d_{\nu+j, 2(\nu+j)-1}, 0, \dots, 0)$ ,  $j \in \mathbb{Z}_\mu$ , form a linearly independent set in  $\mathbb{R}^{\mu+\nu}$ , since, for every  $j \in \mathbb{Z}_\mu$ ,  $D_{\nu+j}$  is of exact degree  $(2(\nu+j)-1)$ , as follows from the explicit formulas (2.16). Hence, the coefficient sequences associated with the Laurent polynomials  $D_{\nu+j}$ ,  $j \in \mathbb{Z}_\mu$ , form a basis for the nullspace  $\mathcal{N}(X)$ , and consequently there exist unique real numbers  $t_j$ ,  $j \in \mathbb{Z}_\mu$ , so that (2.65) holds. Hence, from (2.65), (1.3) and (2.13), we have

$$a_{2k+1} = \sum_{j \in \mathbb{Z}_\mu} t_j d_{\nu+j, 2k+1}, \quad k \in \mathbb{Z}_{\nu+\mu-1},$$

and thus, using (2.68) for  $\ell = 0$ , together with (2.33) and (2.20), we find that

$$\frac{1}{2} = \sum_{k \in \mathbb{Z}_{\nu+\mu-1}} a_{2k+1} = \sum_{j \in \mathbb{Z}_\mu} t_j \sum_{k \in \mathbb{Z}_{\nu+\mu-1}} d_{\nu+j, 2k+1} = \frac{1}{2} \sum_{j \in \mathbb{Z}_\mu} t_j,$$

and thus  $\sum_{j \in \mathbb{Z}_\mu} t_j = 1$ .

Conversely, suppose the Laurent polynomial  $A$  is given by (2.65), where  $\sum_{j \in \mathbb{Z}_\mu} t_j = 1$ . Then (2.65) and (2.20) yield

$$A(z^{-1}) = \sum_{j \in \mathbb{Z}_\mu} t_j D_{\nu+j}(z^{-1}) = \sum_{j \in \mathbb{Z}_\mu} t_j D_{\nu+j}(z) = A(z), \quad z \in \mathbb{C} \setminus \{0\},$$

so that  $A$  satisfies (2.36), whereas

$$A(z) + A(-z) = \sum_{j \in \mathbb{Z}_\mu} t_j [D_{\nu+j}(z) + D_{\nu+j}(-z)] = 2 \sum_{j \in \mathbb{Z}_\mu} t_j = 2, \quad z \in \mathbb{C} \setminus \{0\},$$

from the top line of (2.12), together with Proposition 2.1, thereby showing that  $A$  also satisfies (2.37).

Finally, suppose  $p \in \pi_{2\nu-1}$ . Since (2.65), (1.3) and (2.13) yield

$$a_j = \sum_{k \in \mathbb{Z}_\mu} t_k d_{\nu+k,j}, \quad j \in \mathbb{Z},$$

we get, for  $j \in \mathbb{Z}$ , and using (2.14), that

$$\begin{aligned} \sum_k a_{j-2k} p(k) &= \sum_k \left[ \sum_{\ell \in \mathbb{Z}_\mu} t_\ell d_{\nu+\ell,j-2k} \right] p(k) \\ &= \sum_{\ell \in \mathbb{Z}_\mu} t_\ell \left[ \sum_k d_{\nu+\ell,j-2k} p(k) \right] \\ &= \left[ \sum_{\ell \in \mathbb{Z}_\mu} t_\ell \right] p\left(\frac{j}{2}\right) = p\left(\frac{j}{2}\right), \end{aligned}$$

thereby showing that  $\mathbf{a}$  also satisfies the polynomial filling property (2.38). It follows that  $A \in \mathcal{A}_{\mu,\nu}$ . ■

We now proceed in Propositions 2.8 and 2.9 below to identify two subclasses of  $\mathcal{A}_{\mu,\nu}$  that satisfy the conditions of Theorem 2.2, and for which we are therefore guaranteed the existence of a refinable function and the convergence of the associated subdivision scheme  $S_a$ .

**PROPOSITION 2.8** *For  $\mu \in \mathbb{Z}_+$ ,  $\nu \in \mathbb{N}$ , suppose that the Laurent polynomial  $A$  belongs to the class  $\mathcal{A}_{\mu,\nu}$  and, moreover, is such that, in the Dubuc–Deslauriers expansion (2.65) of*

Theorem 2.7, we have

$$t_j \geq 0, \quad j \in \mathbb{Z}_\mu. \quad (2.69)$$

Then the conditions of Theorem 2.2 are satisfied with  $\mathbf{n} = \nu + \mu$ .

**Proof.** From (2.65), (2.35) and (2.69), we see that  $A$  has the property  $A(z) \geq 0$ ,  $|z| = 1$ , with equality if and only if  $z = -1$ . Hence  $A$  satisfies (2.22) and (2.23). Since  $A$  also satisfies the interpolatory condition (2.37), so that (2.21) holds, we conclude that  $A$  satisfies the conditions of Theorem 2.2 with  $\mathbf{n} = \nu + \mu$ .  $\blacksquare$

For the proof our next result, we are indebted to Tomas Sauer.

**PROPOSITION 2.9** *For  $\mu \in \mathbb{Z}_+$ ,  $\nu \in \mathbb{N}$ , suppose the Laurent polynomial  $A \in \mathcal{A}_{\mu, \nu}$  has a coefficient sequence  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$  in (1.3) satisfying*

$$a_{2j-1} \begin{cases} > 0, & j \in \mathbb{J}_{\nu+\mu}, \\ = 0, & j \notin \mathbb{J}_{\nu+\mu}. \end{cases} \quad (2.70)$$

Then  $A$  satisfies the conditions of Theorem 2.2 with  $\mathbf{n} = \nu + \mu$ .

**Proof.** Since  $A \in \mathcal{A}_{\mu, \nu}$ , it follows from (1.3) and (2.21) that  $A(z) = 1 + \tilde{A}(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ , where

$$\tilde{A}(z) = \sum_{j \in \mathbb{J}_{\nu+\mu}} a_{2j-1} z^{2j-1}, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.71)$$

Since, as noted before in (2.40), the condition (2.38) implies

$$\sum_j a_{2j-1} = 1, \quad (2.72)$$

we deduce from (2.41) that  $\tilde{A}$  has the property  $\tilde{A}(1) = 1$ . Using (2.71) and (2.52), we then obtain, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} \tilde{A}(e^{ix}) &= \sum_{j \in \mathbb{J}_{\mu+\nu}} a_{2j-1} e^{ix(2j-1)} \\ &= \sum_{j=-\mu-\nu+1}^0 a_{2j-1} e^{ix(2j-1)} + \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{2j-1} e^{ix(2j-1)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{-2j+1} e^{-ix(2j-1)} + \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{2j-1} e^{ix(2j-1)} \\
 &= \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{2j-1} e^{-ix(2j-1)} + \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{2j-1} e^{ix(2j-1)} \\
 &= \sum_{j \in \mathbb{N}_{\mu+\nu}} a_{2j-1} [e^{-ix(2j-1)} + e^{ix(2j-1)}] \\
 &= 2 \sum_{j \in \mathbb{N}_{\nu+\mu}} a_{2j-1} \cos(2j-1)x. \tag{2.73}
 \end{aligned}$$

In particular, (2.73) shows that  $\tilde{A}(e^{ix})$  is real for  $x \in \mathbb{R}$ , and according to (2.70), and the symmetry (2.52) of  $\mathbf{a}$ , we conclude that, for  $x \in \mathbb{R}$ , we have

$$|\tilde{A}(e^{ix})| \leq 2 \sum_{j \in \mathbb{N}_{\nu+\mu}} a_{2j-1} = \sum_j a_{2j-1} = 1, \quad x \in \mathbb{R}, \tag{2.74}$$

from (2.72).

Moreover, from (2.73), (2.70) and (2.74),  $x \in \mathbb{R}$  satisfies the equation  $\tilde{A}(e^{ix}) = -1$  if and only if for every  $j \in \mathbb{N}_{\nu+\mu}$  we have that  $\cos(2j-1)x = -1$ , i.e. if and only if  $x = (2k+1)\pi$ ,  $k \in \mathbb{Z}$ . Since also  $A(z) = 1 + \tilde{A}(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ , it then follows that  $A(z) \geq 0$  for  $|z| = 1$ , with equality if and only if  $z = -1$ , thereby proving that  $\mathbf{A}$  satisfies the conditions of Theorem 2.2 with  $\mathbf{n} = \mu + \nu$ . ■

For any given  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ , we can construct a mask symbol  $\mathbf{A} \in \mathcal{A}_{\mu,\nu}$  using the Dubuc–Deslauriers expansion (2.65) of Theorem 2.7 which, moreover, has a convergent corresponding subdivision scheme  $\mathbf{S}_{\mathbf{a}}$  if  $\mathbf{t}_j \geq 0$ ,  $j \in \mathbb{Z}$ , by virtue of Proposition 2.8 or  $\mathbf{a}_j > 0$ ,  $j \in \mathbb{J}_{\mu+\nu}$ , by virtue of Proposition 2.9. In the next section we develop two other construction techniques for  $\mathbf{A} \in \mathcal{A}_{\mu,\nu}$ , which similarly yield convergent interpolatory subdivision schemes  $\mathbf{S}_{\mathbf{a}}$ .

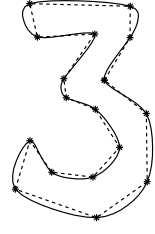




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# Two classes of convergent subdivision schemes from $\mathcal{A}_{\mu,\nu}$

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In this chapter we present two general methods, as were first introduced in [22], of generating a Laurent polynomial  $A \in \mathcal{A}_{\mu,\nu}$  associated with a convergent, symmetric interpolatory subdivision scheme. The first method is based on the sampling at half integers of a given symmetric function  $Q \in C_0(\mathbb{R})$  with a certain polynomial reproducing property, and the second one depends on solving a Bezout identity associated with a given symmetric Hurwitz polynomial  $H$  of even positive degree and with a zero at  $z = -1$ .

## 3.1 The generating function $Q$

Following [22], we start with a continuous function  $Q \in C_0(\mathbb{R})$ , and then define the mask  $a_j \in M_0(\mathbb{Z})$  by

$$a_j = \begin{cases} \delta_j, & j \text{ even}, \\ Q\left(\frac{j}{2}\right), & j \text{ odd}. \end{cases} \quad (3.1)$$

To ensure that the corresponding mask symbol  $A$  belongs to the class  $\mathcal{A}_{\mu,\nu}$ , we demand that  $Q$  satisfies

$$Q(x) = 0, \quad x \notin (-\nu - \mu, \nu + \mu); \quad (3.2)$$

$$Q = Q(-\cdot); \quad (3.3)$$

$$\sum_j p(j) Q(\cdot - j) = p, \quad p \in \pi_{2\nu-1}. \quad (3.4)$$

The following result then holds.

**THEOREM 3.1** *For  $\mu \in \mathbb{Z}_+$ ,  $\nu \in \mathbb{N}$ , suppose the function  $Q \in C_0(\mathbb{R})$  satisfies the three properties (3.2), (3.3) and (3.4). Then, with the mask  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$  defined by (3.1), the associated mask symbol  $\mathbf{A}$  defined by (1.3) belongs to the class  $\mathcal{A}_{\mu,\nu}$ .*

**Proof.** According to (1.3), (3.1) and (3.2), we have that  $\mathbf{A}$  has degree at most  $2(\mu+\nu)-1$ . Moreover, (2.36) and (2.37) are immediate consequences of (3.1) and (3.3). It remains to prove that the polynomial reproduction property (2.38) holds.

Suppose therefore that  $\mathbf{p} \in \pi_{2\nu-1}$ . If  $j = 2\mathbf{n}$ ,  $\mathbf{n} \in \mathbb{Z}$ , we have, from the top line of (3.1),

$$\sum_{\mathbf{k}} \mathbf{a}_{j-2\mathbf{k}} \mathbf{p}(\mathbf{k}) = \sum_{\mathbf{k}} \mathbf{a}_{2\mathbf{n}-2\mathbf{k}} \mathbf{p}(\mathbf{k}) = \mathbf{p}(\mathbf{n}) = \mathbf{p}\left(\frac{j}{2}\right).$$

If  $j = 2\mathbf{n} + 1$ ,  $\mathbf{n} \in \mathbb{Z}$ , we have

$$\sum_{\mathbf{k}} \mathbf{a}_{j-2\mathbf{k}} \mathbf{p}(\mathbf{k}) = \sum_{\mathbf{k}} \mathbf{a}_{2\mathbf{n}+1-2\mathbf{k}} \mathbf{p}(\mathbf{k}) = \sum_{\mathbf{k}} \mathbf{p}(\mathbf{k}) Q(\mathbf{n} + \frac{1}{2} - \mathbf{k}) = \mathbf{p}(\mathbf{n} + \frac{1}{2}) = \mathbf{p}\left(\frac{j}{2}\right),$$

by virtue of (3.4). Hence (2.38) is satisfied. ■

If, in addition to (3.2), (3.3) and (3.4),  $Q$  is a fundamental interpolant, i.e.

$$Q(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{3.5}$$

and is refinable, then it is also a refinable function corresponding to the mask (3.1) defined above. Indeed, if

$$Q = \sum_j \mathbf{q}_j Q(2 \cdot -j),$$

then the condition (3.5) implies  $\mathbf{q}_j = Q(\frac{j}{2})$ ,  $j \in \mathbb{Z}$ , and thus  $\mathbf{q} = \mathbf{a}$  (cf. (2.56) in Theorem 2.6). This observation provides some motivation for the choice of the mask sequence described in (3.1).

In Sections 3.2 and 3.4 below, we present two admissible choices for the function  $Q$ .

### 3.2 Choosing $Q$ as a centered cardinal B-spline

As in [22], our first choice of  $Q$  is the centered cardinal B-spline of even order. The next theorem proves that this choice not only satisfies the conditions (3.2), (3.3) and (3.4), and is therefore valid in the context of Theorem 3.1, but also yields a convergent symmetric interpolatory subdivision scheme.

**THEOREM 3.2** *For  $m \in \mathbb{N}$ , the Laurent polynomial  $A$  defined by (1.3), (3.1), with the choice*

$$Q = N_{2m}(m + \cdot), \quad (3.6)$$

*belongs to the class  $\mathcal{A}_{m-1,1}$  and satisfies the conditions of Theorem 2.2 with  $n = m$ .*

**Proof.** According to Theorem 3.1, it will suffice to prove that the choice (3.6) for  $Q$  satisfies the three conditions (3.2), (3.3) and (3.4), with  $\mu = m - 1$  and  $\nu = 1$ .

From (3.6) and (1.17) we see that (3.2) holds with  $\mu = m - 1$  and  $\nu = 1$ . Also, (1.22) gives

$$N_{2m}(m + \cdot) = N_{2m}(m - \cdot), \quad (3.7)$$

and thus, we see for  $j \in \mathbb{Z}$ , from (3.6), that

$$Q(-\cdot) = N_{2m}(m - \cdot) = N_{2m}(m + \cdot) = Q,$$

and thus  $Q$  satisfies (3.3).

It remains to prove the identity

$$\sum_j j N_{2m}(x + m - j) = x, \quad x \in \mathbb{R}, \quad (3.8)$$

which, together with the fact that, from (1.19),

$$\sum_j N_{2m}(x + m - j) = 1, \quad x \in \mathbb{R},$$

will then prove that (3.4) holds with  $\nu = 1$ .

To this end, we first note, from (1.14) and (1.23) that the function  $F \in C(\mathbb{R})$  defined by

$$F = \sum_j j N_2(\cdot + 1 - j)$$

satisfies  $F \in S_2(\mathbb{R})$  and  $F(k) = k$ ,  $k \in \mathbb{Z}$ , and thus  $F(x) = x$ ,  $x \in \mathbb{R}$ , since the function  $F$  is a piecewise linear polynomial with breakpoints at the integers. Hence (3.8) holds for  $m = 1$ .

If  $m \geq 2$ , we define the function  $G \in S_{2m}(\mathbb{Z})$  by

$$G(x) = \sum_j j N_{2m}(x + m - j) - x, \quad x \in \mathbb{R}. \quad (3.9)$$

Now we use the formula (1.21) for the derivative of a cardinal B-spline to obtain, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} G'(x) &= \sum_j j \left[ N_{2m-1}(x + m - j) - N_{2m-1}(x + m - j - 1) \right] - 1, \\ &= \sum_j j N_{2m-1}(x + m - j) - \sum_j (j - 1) N_{2m-1}(x + m - j) - 1, \\ &= \sum_j N_{2m-1}(x + m - j) - 1 = 0, \end{aligned}$$

from (1.19), and thus

$$G'(x) = 0, \quad x \in \mathbb{R}. \quad (3.10)$$

If we can show that

$$G(0) = 0, \quad (3.11)$$

then it would follow from (3.10) that  $G(x) = 0$ ,  $x \in \mathbb{R}$ , which, together with (3.9), would then imply that (3.8) also holds for  $m \geq 2$  and our proof will be complete. To this end, we now use the definition (3.9) of  $G$ , together with (1.17), and (3.7), to obtain

$$\begin{aligned} G(0) &= \sum_j j N_{2m}(m - j) \\ &= \sum_{j=-m+1}^{m-1} j N_{2m}(m - j) \\ &= \sum_{j=-m+1}^{-1} j N_{2m}(m - j) + \sum_{j \in \mathbb{N}_{m-1}} j N_{2m}(m - j) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j \in \mathbb{N}_{m-1}} (-j) N_{2m}(m+j) + \sum_{j \in \mathbb{N}_{m-1}} j N_{2m}(m-j) \\
 &= - \sum_{j \in \mathbb{N}_{m-1}} j N_{2m}(m-j) + \sum_{j \in \mathbb{N}_{m-1}} j N_{2m}(m-j) = 0,
 \end{aligned}$$

thereby proving the desired result (3.11).

Finally, we appeal to (3.1), (3.6) and (1.18) to deduce that the conditions of Proposition 2.9 are satisfied with  $\mu = m - 1$  and  $\nu = 1$ , from which it then follows that the conditions of Theorem 2.2 are satisfied.  $\blacksquare$

For example, with  $m = 2$ , we have  $Q = N_4(2 + \cdot)$  in (3.6), so that if we choose the mask  $\alpha$  as in (3.1), we obtain the mask symbol

$$A(z) = \frac{1}{48} \left( z^{-3} + 23z^{-1} + 48 + 23z + z^3 \right), \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.12)$$

The associated refinable function  $\phi$  is plotted in Figure 3.1 and the convergent subdivision scheme with mask symbol  $A$  is illustrated in Figure 3.2.

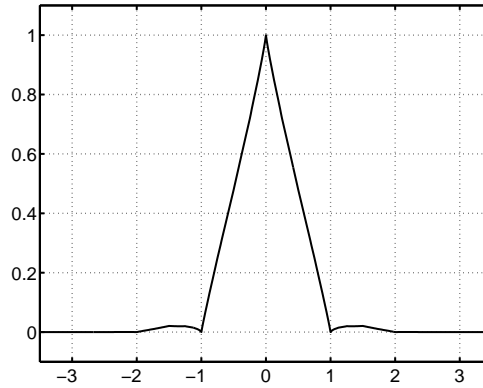


Figure 3.1: The refinable function  $\phi$  with mask symbol (3.12).

In the paper [46, Section 3], the authors proved existence and convergence for the non-interpolatory mask  $\alpha \in M_0(\mathbb{Z})$  defined, according to [46, equation (3.6)], by  $\alpha_j = N_m(\frac{j}{2})$ ,  $j \in \mathbb{Z}$ , which is different from, but related to, the interpolatory mask defined by (3.1), (3.6).

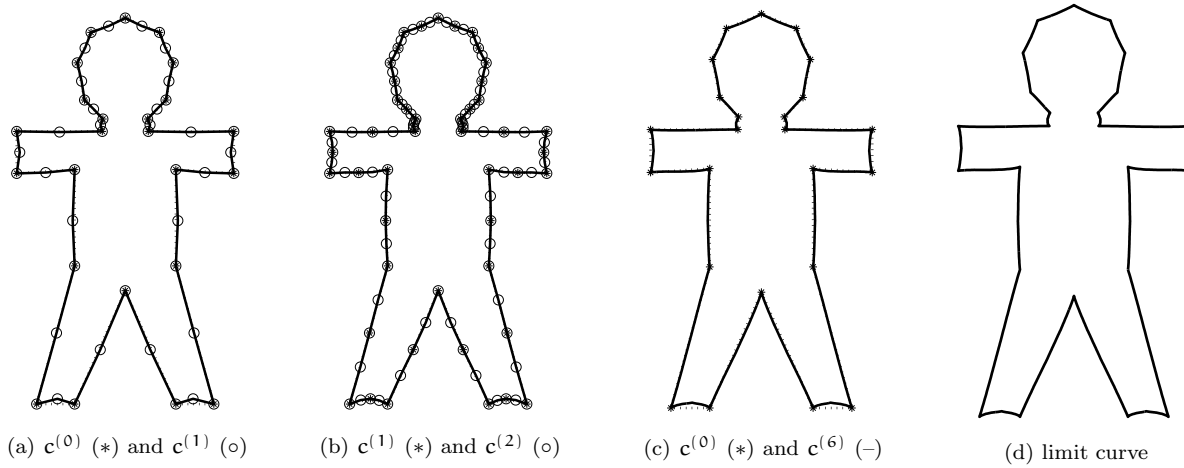


Figure 3.2: Subdivision with mask symbol (3.12).

### 3.3 The generating Hurwitz polynomial $H$

Following [22], our second general method for computing a convergent symmetric interpolatory subdivision scheme in  $\mathcal{A}_{\mu,\nu}$ , is based on a given Hurwitz polynomial  $H$  satisfying certain additional conditions.

We shall rely on the following two propositions from [22] concerning Bezout identities.

**PROPOSITION 3.3** *Suppose, for  $n \in \mathbb{N}$ , that  $H = \sum_{j \in \mathbb{Z}_{2n}} h_j(\cdot)^j$  is a Hurwitz polynomial of even degree  $2n$ . Then there exists a unique polynomial  $G = \sum_{j \in \mathbb{Z}_{2n-2}} g_j(\cdot)^j$  of even degree  $2n-2$ , such that the Bezout identity*

$$H(z)G(z) - H(-z)G(-z) = z^{2n-1}, \quad z \in \mathbb{C}, \quad (3.13)$$

*is satisfied. Moreover, the coefficients of  $G$  have the alternating sign property*

$$(-1)^{n+j+1} g_j > 0, \quad j \in \mathbb{Z}_{2n-2}. \quad (3.14)$$

**Proof.** To prove the existence of a unique polynomial  $G$  in  $\pi_{2n-1}$  satisfying the Bezout identity (3.13), we use a method similar to the one that was employed in the proof of [24, Theorem 5].

Since  $H$  is a Hurwitz polynomial, the two polynomials  $H$  and  $H(-\cdot)$  have no common factors, and thus, according to a standard result in polynomial algebra, there exist two polynomials  $U$  and  $V$  such that

$$H(z)U(z) + H(-z)V(z) = 1, \quad z \in \mathbb{C}. \quad (3.15)$$

Using, for a function  $f: \mathbb{C} \rightarrow \mathbb{C}$ , the notation  $f_-$  for the function  $f_-(z) = f(-z)$ ,  $z \in \mathbb{C}$ , it then follows from (3.15) that  $WHU + WH_-V = W$ , where the polynomial  $W$  is defined by  $W(z) = z^{2n-1}$ ,  $z \in \mathbb{C}$ . Denote by  $Q$  and  $R$  the unique polynomials, with  $R \in \pi_{2n-1}$ , such that  $WV = QH + R$ . But then  $HM + H_-R = W$ , where the polynomial  $M$  is defined by  $M = WU + QH_-$ . Hence  $\deg(M) = \deg(W - H_-R) - \deg(H) \leq (4n-1) - 2n = 2n-1$ , i.e.,  $M \in \pi_{2n-1}$ . Suppose now that  $\widetilde{M}, \widetilde{R} \in \pi_{2n-1}$  are such that  $H\widetilde{M} + H_-\widetilde{R} = W$ , and define  $K = M - \widetilde{M}$  and  $L = \widetilde{R} - R$ , so that  $HK = H_-L$ , with  $K, L \in \pi_{2n-1}$ . Since  $H$  and  $H_-$  have no common factors, and  $H$  has degree  $2n$ , we deduce that  $K = L = 0$ , i.e.,  $M$  and  $R$  are the unique polynomials in  $\pi_{2n-1}$  such that  $HM + H_-R = W$ . Since also  $H_-M_- + HR_- = W_- = -W$ , and thus,  $(H)(-R_-) + (H_-)(-M_-) = W$ , with  $M_-, R_- \in \pi_{2n-1}$ , we deduce that  $M = -R_-$ , or, equivalently,  $R = -M_-$ . Hence, with the definition  $G = -R_-$ , we have shown that  $G$  is the unique polynomial in  $\pi_{2n-1}$  such that (3.13) holds.

Recalling the fact that  $H$  is a polynomial of even degree  $2n$ , we see that (3.13) does not allow  $G$  to have odd degree, hence  $G \in \pi_{2n-2}$ .

Next, to prove (3.14), we first note from (3.13) that the coefficients of  $G$  are determined by solving the linear system

$$\sum_{k \in \mathbb{Z}_{2n-2}} h_{2j+1-k} g_k = \frac{1}{2} \delta_{n-1,j}, \quad j \in \mathbb{Z}_{2n-2}. \quad (3.16)$$

Now define the matrix  $H$  by

$$H = (H_{jk}), \quad H_{jk} = h_{2j+1-k}, \quad j, k \in \mathbb{Z}_{2n-2},$$

which, since  $H$  is a Hurwitz polynomial, and according to the result [50, Lemma 2.4], is invertible.

Using the notation

$$H \begin{pmatrix} \mathbf{r}_1, & \dots, & \mathbf{r}_p \\ \mathbf{c}_1 & \dots, & \mathbf{c}_p \end{pmatrix}, \quad p \in \mathbb{N}_{2n-2},$$

for the determinant of the submatrix of  $H$  formed by rows  $\mathbf{r}_1, \dots, \mathbf{r}_p$  and columns  $\mathbf{c}_1, \dots, \mathbf{c}_p$ , as well as the definition  $\{\mathbf{k} + \mathbb{N}_n\} = \{\mathbf{k} + 1, \mathbf{k} + 2, \dots, \mathbf{k} + n\}$  for  $n \in \mathbb{N}$  and  $\mathbf{k} \in \mathbb{Z}$ , we next apply Cramer's rule to (3.16) to obtain the formula

$$g_k = (-1)^{n+k+1} \frac{1}{2} \frac{H \begin{pmatrix} \mathbf{r}_1, \dots, \mathbf{r}_{2n-2} \\ \mathbf{c}_{k,1}, \dots, \mathbf{c}_{k,2n-2} \end{pmatrix}}{\det(H)}, \quad k \in \mathbb{N}_{2n-2}, \quad (3.17)$$

where

$$\mathbf{r}_\ell = \begin{cases} \ell, & \ell \in \mathbb{N}_{n-1}, \\ \ell + 1 & \ell \in \{\mathbf{n} - 1 + \mathbb{N}_{n-1}\}, \end{cases}$$

and

$$\mathbf{c}_{k,\ell} = \begin{cases} \ell, & \ell \in \mathbb{N}_k \\ \ell + 1 & \ell \in \{\mathbf{k} + \mathbb{N}_{2n-2-k}\}. \end{cases}$$

Specialising the result [50, Lemma 2.4], we conclude that the two determinants in (3.17) are positive, thereby proving the desired result (3.14), as well as the fact that  $\deg(G) = 2n - 2$ . ■

We shall say that a polynomial  $p$  of degree  $n$ , as defined by  $p(z) = \sum_{j \in \mathbb{Z}_n} p_j z^j$ ,  $z \in \mathbb{C}$ , is a **symmetric polynomial** if  $p_{n-j} = p_j$ ,  $j \in \mathbb{Z}_n$ , or, equivalently, if  $z^n p(z^{-1}) = p(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ . Since, for a symmetric polynomial  $p$  with  $\deg(p) = n$ , we must have  $p(0) \neq 0$ , we deduce that the zeros of a symmetric polynomial  $p$  occur in reciprocal pairs, i.e. for  $z_0 \in \mathbb{C} \setminus \{0\}$  we have  $p(z_0) = 0$  if and only if  $p(z_0^{-1}) = 0$ . Hence, if a symmetric polynomial  $p$  of degree  $n$  is such that  $p(-1) = 0$  and  $p(1) \neq 0$ , and if we denote the order of zero at  $z = -1$  by  $\nu \in \mathbb{N}$ , then it must be true that  $n$  and  $\nu$  are either both even, or both odd.

The following specialisation of Proposition 3.3 can now be proved.

**PROPOSITION 3.4** *Suppose, in Proposition 3.3, we choose  $H$  as a symmetric polynomial with only negative zeros, and with  $H(-1) = 0$ . Then  $G$  is a symmetric polynomial satisfying*

$$G(z) \neq 0, \quad |z| = 1. \quad (3.18)$$



**Proof.** From the Bezout identity (3.13), as well as the symmetry of the polynomial  $H$ , we have, for  $z \in \mathbb{C} \setminus \{0\}$ , that

$$\begin{aligned} H(z) \left[ z^{2n-2} G(z^{-1}) \right] - H(-z) \left[ (-z)^{2n-2} G(-z^{-1}) \right] \\ = z^{4n-2} \left[ H(z^{-1}) G(z^{-1}) - H(-z^{-1}) G(-z^{-1}) \right] \\ = z^{4n-2} z^{-2n+1} = z^{2n-1}, \end{aligned}$$

and thus, from the uniqueness result in Proposition 3.3, it follows that  $z^{2n-2} G(z^{-1}) = G(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ , i.e.  $G$  is a symmetric polynomial.

Noting in particular from (3.14) that

$$(-1)^n G(-1) > 0, \quad (3.19)$$

we finally refer to [37, Lemmas 2.3 and 2.4] (see also [38, Theorem 4.3]) for the proof of (3.18). ■

The result of Proposition 3.4 allows us, as was done in [22], to prove the following theorem.

**THEOREM 3.5** *In Proposition 3.4, suppose that, for  $\nu \in \mathbb{N}_n$ , the polynomial  $H$  has a zero of order  $2\nu$  at  $z = -1$ . Then the Laurent polynomial  $A$  defined by*

$$A(z) = 2z^{-2n+1} H(z) G(z), \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.20)$$

*belongs to the class  $\mathcal{A}_{n-\nu,\nu}$ , and satisfies the conditions of Theorem 2.2.*

**Proof.** First, note from (3.20) that  $\deg(A) = 2n - 1$ . Moreover, the symmetry of the polynomials  $H$  and  $G$  ensure that the Laurent polynomial  $A$  in (3.20) is symmetric in the sense of (2.36).

Next we observe from (3.20) and (3.13) that  $A$  also satisfies the interpolatory condition (2.37). Since  $H$  has a zero of order  $2\nu$  at  $z = -1$ , it follows from (3.20) and (3.19) that  $A$  also satisfies the condition (2.42), so that we can deduce from Proposition 2.5 that the polynomial filling property (2.38) holds. Hence  $A \in \mathcal{A}_{n-\nu,\nu}$ .

Finally, we conclude from (3.20), (2.41), together with the hypothesis that  $H$  has only negative zeros, and (3.18), that  $A$  satisfies the positivity condition (2.23). Hence  $A$  satisfies all the conditions of Theorem 2.2.  $\blacksquare$

**Remark:** Observe that the case  $\nu = n$  in Theorem 3.5 yields the Hurwitz polynomial

$$H(z) = (1 + z)^{2n}, \quad z \in \mathbb{C},$$

with the resulting Laurent polynomial  $A$ , as defined by (3.20), belonging to the class  $\mathcal{A}_{0,n}$ , so that, from (2.39), we have here that  $A = D_n$ , the Dubuc–Deslauriers mask symbol as given explicitly by (2.13), (2.16). Theorem 3.5 therefore provides an alternative proof to the one in Theorem 2.3, as based on (2.34) and (2.35), to show that  $A = D_n$  satisfies the positivity condition (2.23), and therefore also all the conditions of Theorem 2.2.

As a second example we choose  $n = 2$  and the Hurwitz polynomial  $H$  in Theorem 3.5 as

$$H(z) = (1 + z)^2(2 + z)(1 + 2z), \quad z \in \mathbb{C},$$

so that  $\nu = 1$ . The unique polynomial  $G$  of degree 2 satisfying the Bezout identity (3.13) is then given by

$$G(z) = -\frac{1}{180}(2 - 9z + 2z^2), \quad z \in \mathbb{C},$$

and the resulting Laurent polynomial  $A \in \mathcal{A}_{1,1}$ , according to (3.20), has the formulation

$$A(z) = \frac{1}{90}(-4z^{-3} + 49z^{-1} + 90 + 49z - 4z^3), \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.21)$$

The corresponding refinable function  $\phi$  of Theorem 2.2 is plotted in Figure 3.3(a) and the associated convergent subdivision scheme  $S_a$  is illustrated in Figure 3.4.

Note that, although the refinable function  $\phi$  in Figure 3.4(a) appears to satisfy  $\phi(x) = 0$ ,  $x \notin (-2, 2)$ , we see from (2.24) in Theorem 2.2, that  $\phi(x) = 0$ ,  $x \notin (-3, 3)$  and from (2.57) in Theorem 2.6, together with (2.62), that  $\phi$  has clustered zeros towards the boundaries of its support interval  $(-3, 3)$ , as illustrated in Figure 3.4(b) and (c).

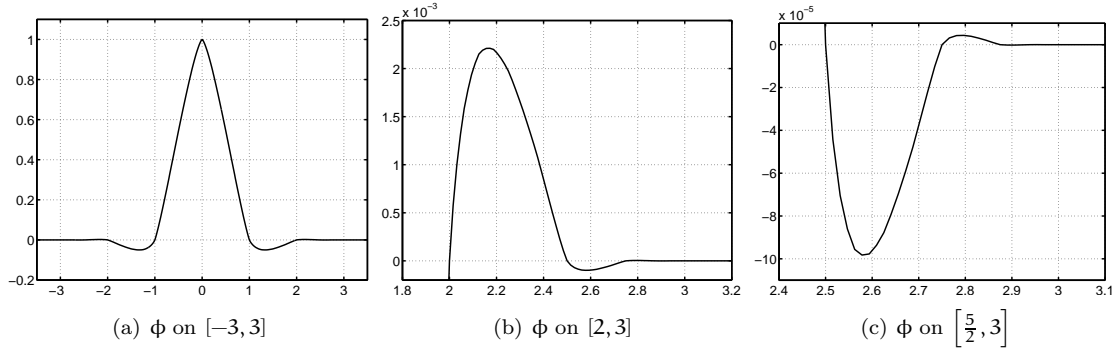
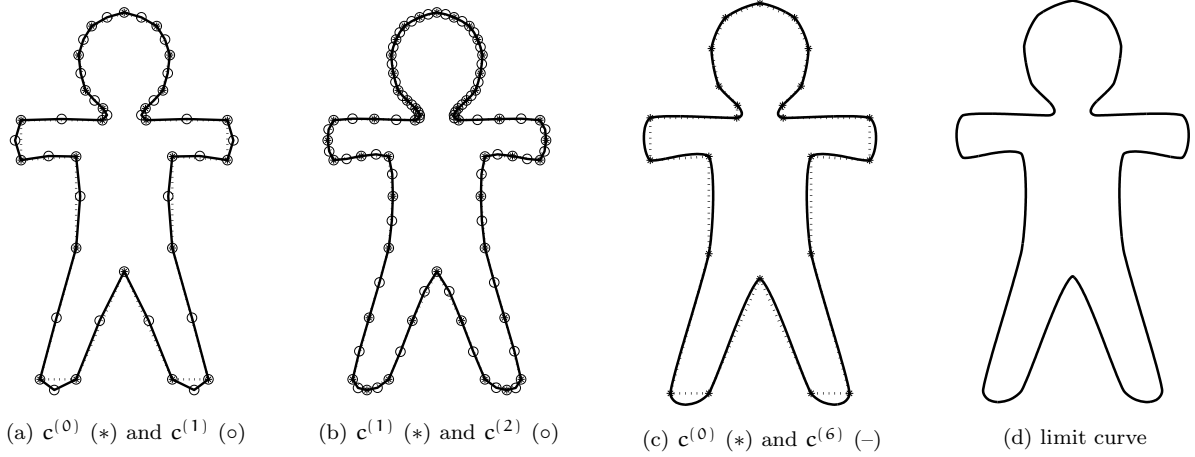

 Figure 3.3: The refinable function  $\phi$  with mask symbol (3.21)


Figure 3.4: Subdivision with the mask symbol (3.21)

In [46], the authors considered the mask symbol  $A$  defined by (3.20), with, for  $M \in \mathbb{Z}_+$  and an integer  $m \geq 3$ , the specific choice  $H(z) = 2 \left(\frac{1+z}{2}\right)^{m+2M} E_m(z)$ ,  $z \in \mathbb{C}$ , in (3.13), where  $E_m$  is the Euler–Frobenius polynomial of degree  $m - 2$ , as defined (see e.g. [61, Theorem 2.1]) by

$$E_m(z) = \sum_{j \in \mathbb{Z}_{m-2}} N_m(j+1) z^j, \quad z \in \mathbb{C}, \quad (3.22)$$

where, from (1.13), we have

$$N_m(j+1) = \frac{1}{(m-1)!} \sum_{k \in \mathbb{Z}_j} (-1)^k \binom{m}{k} (j+1-k)^{m-1}, \quad j \in \mathbb{Z}_{m-2} \quad (m \geq 2). \quad (3.23)$$

Our approach has the advantage of also proving the existence of the corresponding refinable function  $\phi \in C(\mathbb{R})$ .

In the following section, we choose  $Q$  as a fundamental interpolant and then show that

this construction technique is equivalent to a specific choice of  $H$  in (3.13).

## 3.4 Choosing $Q$ as a fundamental interpolant

In this section, we present the theory developed in [22] to construct, in addition to the centered B-spline case of Section 3.2, a second choice for the generating function  $Q$  of Section 3.1. The cardinal spline special case of the interpolatory mask symbol  $A$  in Theorem 3.10 below was also considered in [46, equation (3.27)]. Our results below extend, simplify and improve upon those in [46]. In particular, we show here, as was already also pointed out in [23], that the strong hypothesis of the existence of a solution to the refinement equation [46, equation (3.37)] can be removed.

Using the results of Section 3.3, we shall show that, starting with a given Hurwitz polynomial (mask symbol)  $B$ , and its associated refinable function  $\varphi$  according to Theorem 1.5, we can construct a fundamental interpolant  $Q \in C_0(\mathbb{R})$  in the linear space spanned by the translates  $\{\varphi(2 \cdot -j) : j \in \mathbb{Z}\}$ . This function  $Q$  is shown to be an admissible choice in the context of Theorem 3.1, and such that the interpolatory condition (3.5) also holds. Moreover, we shall demonstrate that the resulting Laurent polynomial  $A$  constructed from (3.1) and (1.3) then satisfies the conditions of Theorem 2.2.

Recall that a Pólya frequency sequence  $\mathbf{a}$  of all orders is such that *all* the minors of the bi-infinite matrix  $W$  defined by

$$(W)_{jk} = \mathbf{a}_{k-j}, \quad j, k \in \mathbb{Z},$$

are nonnegative. A complete characterisation of such sequences is available in terms of the zeros of the Laurent polynomial  $A = \sum_j \mathbf{a}_j(\cdot)^j$  (see [44, Chapter 8, Theorem 9.5]).

For a Laurent polynomial (or polynomial)  $p = \sum_j p_j(\cdot)^j$ , we shall use the notation  $p^{(e)} = \sum_j p_{2j}(\cdot)^{2j}$ , and  $p^{(o)} = \sum_j p_{2j+1}(\cdot)^{2j+1}$ , to denote the even and odd parts of  $p$ . Note that then  $p^{(e)} = \frac{1}{2}[p + p(-\cdot)]$ , and  $p^{(o)} = \frac{1}{2}[p - p(-\cdot)]$ .

**PROPOSITION 3.6** *Suppose for an integer  $m \geq 2$  that the polynomial  $B = \sum_j b_j(\cdot)^j \in \pi_m$  is a Hurwitz polynomial of degree  $m$ , with  $B(1) = 2$  and  $B(-1) = 0$ . Then there exists a refinable function  $\varphi \in C(\mathbb{R})$  such that*

$$\varphi(x) = 0, \quad x \notin (0, m); \quad (3.24)$$

$$\varphi = \sum_{j \in \mathbb{Z}_m} b_j \varphi(2 \cdot -j) = \sum_j b_j \varphi(2 \cdot -j); \quad (3.25)$$

$$\varphi(x) > 0, \quad x \in (0, m); \quad (3.26)$$

$$\sum_j \varphi(x - j) = 1, \quad x \in \mathbb{R}; \quad (3.27)$$

$$|\varphi(x) - \varphi(y)| \leq K |x - y|^\beta, \quad x, y \in \mathbb{R}, \quad (3.28)$$

for some constant  $K > 0$ , and where  $\beta = -\log_2 \rho$ , with  $\rho = \rho(a)$  as in (1.37) and (1.38), so that  $\beta \in (0, 1]$ .

Moreover, the polynomial  $E$  of degree  $m - 2$  defined by

$$E(z) = \sum_{j \in \mathbb{Z}_{m-2}} \varphi(j+1)z^j = \sum_j \varphi(j+1)z^j, \quad z \in \mathbb{C}, \quad (3.29)$$

has only negative zeros and satisfies the Bezout identity

$$B(z)E(z) - B(-z)E(-z) = 2zE(z^2), \quad z \in \mathbb{C}. \quad (3.30)$$

**Proof.** Since  $B$  is a Hurwitz polynomial of degree  $m$  and  $B(1) > 0$ , it follows that  $b_j > 0$ ,  $j \in \mathbb{Z}_m$ . For the existence of a refinable function  $\varphi \in C(\mathbb{R})$  satisfying the properties (3.24)–(3.27), we can therefore refer to Theorem 1.5. The Hölder continuity property (3.28) of  $\varphi$  is proved in [50, Theorem 2.5 and Corollary 2.1] (see also [20, Theorem 1.2]).

To prove that the polynomial  $E$  defined by (3.29) has only negative zeros, we first appeal to [36, Theorem 4.1 and Corollary 3.1] (see also [50, Theorem 2.6]) to conclude that  $\{\varphi(j) : j \in \mathbb{Z}\}$  is a Pólya frequency sequence of all orders. The fact that  $E$  has only negative zeros then follows directly from [44, Chapter 8, Corollary 3.1].

Finally, to prove the polynomial identity (3.30), we use (3.29) and (3.25) to obtain, for  $z \in \mathbb{C}$ ,

$$\begin{aligned}
 z E(z^2) &= z \sum_j \varphi(j+1) z^{2j} \\
 &= z \sum_j \left[ \sum_k b_k \varphi(2(j+1) - k) \right] z^{2j} \\
 &= z \sum_j \left[ \sum_k b_{2k} \varphi(2j+2-2k) + \sum_k b_{2k+1} \varphi(2j+2-2k-1) \right] z^{2j} \\
 &= z \sum_k b_{2k} \left[ \sum_j \varphi(2j-2k+2) z^{2j} \right] + z \sum_k b_{2k+1} \left[ \sum_j \varphi(2j-2k+1) z^{2j} \right] \\
 &= z \sum_k b_{2k} \left[ \sum_j \varphi(2j+2) z^{2j+2k} \right] + z \sum_k b_{2k+1} \left[ \sum_j \varphi(2j+1) z^{2j+2k} \right] \\
 &= \left[ \sum_k b_{2k} z^{2k} \right] \left[ \sum_j \varphi((2j+1)+1) z^{2j+1} \right] \\
 &\quad + \left[ \sum_k b_{2k+1} z^{2k+1} \right] \left[ \sum_j \varphi(2j+1) z^{2j} \right] \\
 &= B^{(e)}(z) E^{(o)}(z) + B^{(o)}(z) E^{(e)}(z) \\
 &= \frac{1}{2} [B(z) + B(-z)] \frac{1}{2} [E(z) - E(-z)] + \frac{1}{2} [B(z) - B(-z)] \frac{1}{2} [E(z) + E(-z)] \\
 &= \frac{1}{2} [B(z)E(z) - B(-z)E(-z)]. \quad \blacksquare
 \end{aligned}$$

Note that the choice  $B(z) = 2^{-m+1}(1+z)^m$ ,  $z \in \mathbb{C}$ , in Proposition 3.6 gives, according to (1.15), (1.16) and (1.41),  $\varphi = N_m$ , and observe also that the polynomial  $E$  as defined by (3.29) is then the Euler–Frobenius polynomial as given by (3.22).

The fundamental interpolant  $Q \in C_0(\mathbb{R})$  can now be constructed as follows.

**THEOREM 3.7** *Let, for a given integer  $m \geq 2$ , the polynomial  $B$  and the function  $\varphi$  be as defined in Proposition 3.6, and suppose the integer  $\rho \in \mathbb{N}_m$  denotes the order of the zero at  $z = -1$  of  $B$ . Then there exists a sequence  $c \in M_0(\mathbb{Z})$  such that the function  $Q$  defined by*

$$Q = \sum_j c_j \varphi(2 \cdot -j) = \sum_{j=-2m+2}^{m-2} c_j \varphi(2 \cdot -j) \quad (3.31)$$

has the following properties:

$$Q(x) = 0, \quad x \notin (-m+1, m-1); \quad (3.32)$$

$$Q(j) = \delta_j, \quad j \in \mathbb{Z}; \quad (3.33)$$

$$\sum_j p(j)Q(\cdot - j) = p, \quad p \in \pi_{p-1}. \quad (3.34)$$

**Proof.** First, observe that (3.32) is a consequence of (3.31) and (3.24). Next, in the settings of Propositions 3.6 and 3.3, we define the polynomial  $G$  of degree  $2m-4$  as the solution of the Bezout identity (3.13) with  $n = m-1$ , and  $H$  chosen as the Hurwitz polynomial of degree  $2m-2$  as given by

$$H = \frac{1}{2} B E, \quad (3.35)$$

where  $E$  is the polynomial defined by (3.29), and which, according to Proposition 3.6, has only negative zeros. If we now define the Laurent polynomial  $C$  by

$$\sum_j c_j z^j = \sum_{j=-2m+2}^{m-2} c_j z^j = C(z) = z^{-2m+2} B(z) G(z), \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.36)$$

then (3.13) implies the identity

$$z \left[ C(z) E(z) - C(-z) E(-z) \right] = 2, \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.37)$$

We proceed to show that, with the coefficient sequence  $\{c_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$  as chosen in (3.36), the function  $Q$  defined by (3.31) satisfies the interpolatory property (3.33).

We shall, in fact, prove that

$$\sum_j Q(j) z^{2j} = 1, \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.38)$$

which is equivalent to (3.33). To this end, we use (3.31), the definitions (3.36) and (3.29) to obtain, for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$\begin{aligned} \sum_j Q(j) z^{2j} &= \sum_j \sum_k c_k \varphi(2j - k) z^{2j} \\ &= \sum_j \left[ \sum_k c_{2k} \varphi(2j - 2k) + \sum_k c_{2k+1} \varphi(2j - 2k - 1) \right] z^{2j} \end{aligned}$$

$$\begin{aligned}
 &= \sum_k c_{2k} \left[ \sum_j \varphi(2j - 2k) z^{2j} \right] + \sum_k c_{2k+1} \left[ \sum_j \varphi(2j - 2k - 1) z^{2j} \right] \\
 &= \sum_k c_{2k} \left[ \sum_j \varphi(2j + 2) z^{2j+2} \right] z^{2k} + \sum_k c_{2k+1} \left[ \sum_j \varphi(2j + 1) z^{2j+1} \right] z^{2k+1} \\
 &= z \left[ \sum_k c_{2k} z^{2k} \right] \left[ \sum_j \varphi(2j + 2) z^{2j+1} \right] + z \left[ \sum_k c_{2k+1} z^{2k+1} \right] \left[ \sum_j \varphi(2j + 1) z^{2j} \right] \\
 &= z C^{(e)}(z) E^{(o)}(z) + z C^{(o)}(z) E^{(e)}(z) \\
 &= \frac{z}{4} [C(z) + C(-z)] [E(z) - E(-z)] + \frac{z}{4} [C(z) - C(-z)] [E(z) + E(-z)] \\
 &= \frac{z}{2} [C(z)E(z) - C(-z)E(-z)] = 1,
 \end{aligned}$$

by virtue of (3.37), which then implies the desired result (3.38).

With the Fourier transform of a function  $f \in C_0(\mathbb{R})$  defined by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-ixt} f(t) dt, \quad x \in \mathbb{R},$$

we prove property (3.34) by first using the Fourier transform formulations

$$\widehat{Q}(x) = \frac{1}{2} C(e^{-ix/2}) \widehat{\varphi}\left(\frac{x}{2}\right), \quad x \in \mathbb{R},$$

and

$$\widehat{\varphi}(x) = \frac{1}{2} B(e^{-ix/2}) \widehat{\varphi}\left(\frac{x}{2}\right), \quad x \in \mathbb{R}, \quad (3.39)$$

of, respectively, (3.31) and (3.25), as well (3.36), to deduce that

$$\widehat{Q}(x) = e^{i(m-1)x} G(e^{-ix/2}) \widehat{\varphi}(x), \quad x \in \mathbb{R}. \quad (3.40)$$

Exploiting the fact that the polynomial  $B$  has a zero of order  $\rho$  at  $z = -1$ , we use (3.39) inductively, to conclude that  $\widehat{\varphi}^{(\ell)}(2\pi j) = 0$ ,  $j \in \mathbb{Z} \setminus \{0\}$ ,  $\ell \in \mathbb{Z}_{\rho-1}$ , and thus from (3.40),

$$\widehat{Q}^{(\ell)}(2\pi j) = 0, \quad j \in \mathbb{Z} \setminus \{0\}, \quad \ell \in \mathbb{Z}_{\rho-1}. \quad (3.41)$$

Now, for a fixed  $x \in \mathbb{R}$  and  $\ell \in \mathbb{Z}_{\rho-1}$ , define the function  $u$  by  $u(t) := t^\ell Q(x - t)$ ,  $t \in \mathbb{R}$ . It follows that  $u \in C_0(\mathbb{R})$ . Now observe from (3.31) and (3.28) that the function  $Q$ , and therefore also the function  $u$ , is Hölder continuous with exponent  $\beta \in (0, 1]$ , and



therefore also of bounded variation. Hence we can apply the Poisson summation formula

$$\sum_j u(j) = \sum_j \hat{u}(2\pi j), \text{ (see e.g. [8, Chapter 2])}, \text{ together with (3.41), to obtain}$$

$$\sum_j j^\ell Q(x - j) = \sum_{k \in \mathbb{Z}_\ell} (-i)^k \binom{\ell}{k} x^{\ell-k} \hat{Q}^{(k)}(0). \quad (3.42)$$

It follows from (3.42) that the function  $w$  defined by  $w := \sum_j j^\ell Q(\cdot - j)$  is in  $\pi_{\rho-1}$  and satisfies  $w(k) = k^\ell$ ,  $k \in \mathbb{Z}$ , from (3.33). Thus  $w(x) = x^\ell$ ,  $x \in \mathbb{R}$ , thereby proving (3.34).  $\blacksquare$

Next we show that, with the additional constraints of symmetry and real zeros on the Hurwitz polynomial  $B$ , the fundamental function  $Q$  in Theorem 3.7 is also symmetric, and can therefore be used in (3.1) to construct a mask  $A \in \mathcal{A}_{\mu,\nu}$ , for certain  $\mu, \nu \in \mathbb{N}$ . First, we prove the following specialisation of Proposition 3.6.

**PROPOSITION 3.8** *In Proposition 3.6, if it also holds that  $B$  is a symmetric polynomial, then  $\varphi$  is a symmetric function in the sense that  $\varphi(m - \cdot) = \varphi$  and, moreover,  $E$  is a symmetric polynomial of degree  $m - 2$ . Also,*

$$E(-1) \begin{cases} \neq 0, & m \text{ even}, \\ = 0, & m \text{ odd}. \end{cases} \quad (3.43)$$

**Proof.** Define  $\psi \in C_0(\mathbb{R})$  by  $\psi = \varphi(m - \cdot)$ . Then, since  $b_{m-j} = b_j$ ,  $j \in \mathbb{Z}_m$ , by virtue of the fact that  $B$  is a symmetric polynomial of degree  $m$ , we have, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} \sum_j b_j \psi(2x - j) &= \sum_j b_j \varphi(m - 2x + j) \\ &= \sum_j b_{m-j} \varphi(m - 2x + j) \\ &= \sum_j b_j \varphi(2(m - x) - j) \\ &= \varphi(m - x) = \psi(x), \end{aligned} \quad (3.44)$$

by virtue also of the refinability (3.25) of  $\varphi$ . Since, from (3.27),

$$\sum_j \psi(x - j) = \sum_j \varphi(m - x + j) = 1, \quad x \in \mathbb{R},$$

we deduce from (3.44), together with the uniqueness result in Theorem 1.5, that  $\psi = \varphi$ , and thus  $\varphi(\mathfrak{m} - \cdot) = \varphi$ .

If we write  $E(z) = \sum_{j \in \mathbb{Z}_{\mathfrak{m}-2}} e_j z^j$ ,  $z \in \mathbb{C}$ , then, from (3.29), we have

$$e_j = \varphi(j+1), \quad j \in \mathbb{Z}_{\mathfrak{m}-2}, \quad (3.45)$$

and it then follows from the symmetry of  $\varphi$ , as proved above, that

$$e_{\mathfrak{m}-2-j} = \varphi(\mathfrak{m} - 1 - j) = \varphi(j+1) = e_j, \quad j \in \mathbb{Z}_{\mathfrak{m}-2},$$

thereby showing that  $E$  is a symmetric polynomial.

Finally, to prove (3.43), and following a method introduced in [23] and further developed in [22], we set  $z = i$  in the Bezout identity (3.30), and use the fact that both  $B$  and  $E$  are symmetric polynomials, to deduce that

$$\begin{aligned} 2iE(-1) &= B(i)E(i) - B(-i)E(-i) \\ &= B(i)E(i) - B\left(\frac{1}{i}\right)E\left(\frac{1}{i}\right) \\ &= B(i)E(i) - i^{\mathfrak{m}}B(i)i^{\mathfrak{m}-2}E(i) \\ &= [1 + (-1)^{\mathfrak{m}}]B(i)E(i). \end{aligned} \quad (3.46)$$

Since  $B$  is a Hurwitz polynomial, and since, from Proposition 3.6, the polynomial  $E$  has only negative zeros, we see that  $B(i)E(i) \neq 0$ , which together with (3.46), then implies the desired result (3.43). ■

The result of Proposition 3.8 enables us to prove the following.

**THEOREM 3.9** *In Theorem 3.7, if we choose  $B$  as a symmetric polynomial of degree  $\mathfrak{m}$  and with only negative zeros, then the sequence  $\mathbf{c} \in \mathcal{M}_0(\mathbb{Z})$  in (3.31) is symmetric in the sense that*

$$c_{j-2\mathfrak{m}+2} = c_{\mathfrak{m}-2-j}, \quad j \in \mathbb{Z}, \quad (3.47)$$

*and the function  $Q$  defined by (3.31) has the additional property of symmetry, in the sense of (3.3).*

**Proof.** First, we note from (3.35) and (3.36) in the proof of Theorem 3.7 that, with the additional constraints on  $B$  as in this theorem, and since, from Proposition 3.8, we know that the polynomial  $E$  is symmetric and has only negative zeros,  $H$  is a symmetric Hurwitz polynomial of degree  $2m - 2$  with only negatives zeros. But then, recalling also from Proposition 3.3 and 3.4 that then  $G$  is a symmetric polynomial of degree  $2m - 4$ , we find that (3.36) gives, for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$C(z) = z^{-2m+2} \left( z^m B(z^{-1}) \right) \left( z^{2m-4} G(z^{-1}) \right) = z^{-m} \left( z^{2m-2} B(z^{-1}) G(z^{-1}) \right) = z^{-m} C(z^{-1}),$$

thereby proving that  $c_j = c_{-j-m}$ ,  $j \in \mathbb{Z}$ , which is equivalent to the desired result (3.47).

Next, recalling from Proposition 3.8 that  $\varphi(m - \cdot) = \varphi$ , we now use (3.31) and (3.47) to deduce that, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} Q(-x) &= \sum_{j=-2m+2}^{m-2} c_j \varphi(-2x - j) \\ &= \sum_{j \in \mathbb{Z}_{3m-4}} c_{j-2m+2} \varphi(-2x - j + 2m - 2) \\ &= \sum_{j \in \mathbb{Z}_{3m-4}} c_{m-2-j} \varphi(-2x - j + 2m - 2) \\ &= \sum_{j=-2m+2}^{m-2} c_j \varphi(-2x + m + j) \\ &= \sum_{j=-2m+2}^{m-2} c_j \varphi(m - (2x - j)) \\ &= \sum_{j=-2m+2}^{m-2} c_j \varphi(2x - j) = Q(x). \end{aligned}$$

Hence the function  $Q$  is symmetric in the sense of (3.3). ■

An important special case of Theorem 3.9 is provided by the choice  $B(z) = 2 \left( \frac{1+z}{2} \right)^m$ ,  $z \in \mathbb{C}$ , so that  $\rho = m$ , in which case, according to (3.25), Theorem 1.5, (1.15) and (1.16), we have that  $\varphi = N_m$ , the cardinal B-spline of order  $m$ , and  $E = E_m$  is the corresponding Euler–Frobenius polynomial of degree  $m - 2$ , as given by (3.22). The resulting minimally supported fundamental spline interpolant  $Q = Q_m$ , as previously

investigated by various authors in [58], [15], [49], [19, 26, 25, 27], [10] and [38], can therefore be explicitly constructed by solving for the polynomial  $G_m$  of degree  $2m - 4$  from the Bezout identity

$$\left(\frac{1+z}{2}\right)^m E_m(z) G_m(z) - \left(\frac{1-z}{2}\right)^m E_m(-z) G_m(-z) = z^{2m-3}, \quad z \in \mathbb{C}, \quad (3.48)$$

and then defining

$$Q_m = \sum_{j=-2m+2}^{m-2} c_{m,j} N_m(2 \cdot -j), \quad (3.49)$$

where, in the notation  $C_m(z) = \sum_{j=-2m+2}^{m-2} c_{m,j} z^j$ ,  $z \in \mathbb{C} \setminus \{0\}$ , we have

$$C_m(z) = 2z^{-2m+2} \left(\frac{1+z}{2}\right)^m G_m(z), \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.50)$$

These fundamental spline interpolants  $Q_m$  are useful in approximation theory, in the sense that the corresponding local spline interpolation operator  $\mathcal{Q} : M(\mathbb{R}) \rightarrow S_m(\mathbb{Z}/2)$  defined by

$$(\mathcal{Q}f)(x) = \sum_j f(j) Q_m(x - j), \quad x \in \mathbb{R},$$

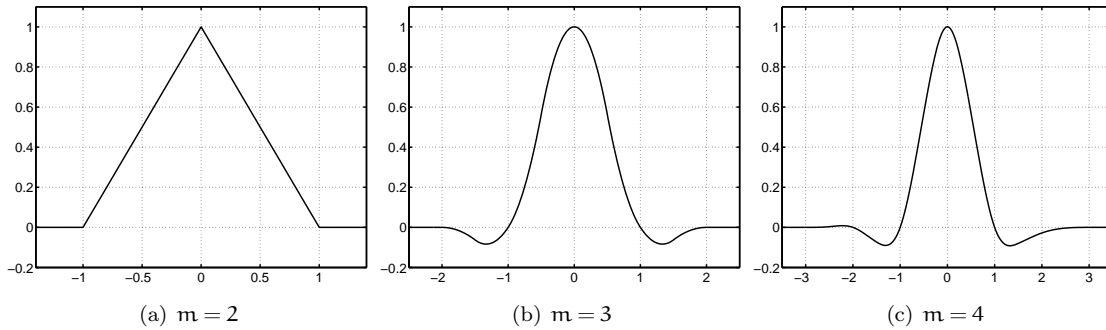
not only has the interpolatory property  $(\mathcal{Q}f)(j) = f(j)$ ,  $j \in \mathbb{Z}$ ,  $f \in M(\mathbb{R})$ , but also, according to (3.34), is exact on the polynomial space  $\pi_{m-1}$ . For  $m = 2, 3, 4$ , the Laurent polynomial  $C_m$  is explicitly calculated by first using (1.13) and (3.23) to obtain

$$E_m(z) = \begin{cases} 1, & m = 2, \\ \frac{1}{2}(1+z), & m = 3, \\ \frac{1}{6}(1+4z+z^2), & m = 4, \end{cases} \quad z \in \mathbb{C}, \quad (3.51)$$

(cf. also [8, p. 191]) and then using (3.48) and (3.50) to find, for  $z \in \mathbb{C}$ , that

$$C_m(z) = \begin{cases} \frac{1}{2}(z^{-2} + 2z^{-1} + 1), & m = 2, \\ \frac{1}{8}(-z^{-4} + z^{-3} + 8z^{-2} + 8z^{-1} + 1 - z), & m = 3, \\ \frac{1}{48}(z^{-6} - 4z^{-5} - 6z^{-4} + 28z^{-3} + 58z^{-2} + 28z^{-1} - 6 - 4z + z^2), & m = 4. \end{cases}$$

Using also (3.49), we can then compute  $Q_m$  for  $m = 2, 3, 4$ , as illustrated in Figure 3.5.


 Figure 3.5: Examples of the fundamental interpolant  $Q_m$ .

**Remark:** The graphs in Figure 3.5 were generated using Lane–Riesenfeld subdivision, as given by (1.24), with initial sequence  $\{c_{m,j} : j \in \{-2m+2, \dots, m-2\}\}$  for, respectively,  $m = 2, 3, 4$ . From Theorem 1.4, we then conclude that this subdivision scheme converges, at a geometric rate, to the limit curve  $\Phi_m = \sum_j c_{m,j} N_m(\cdot - j)$ . Hence, we obtain the desired function  $Q_m$  in (3.49) by setting  $Q_m = \Phi_m(2\cdot)$ .

Our main result of this section is now as follows.

**THEOREM 3.10** *For an integer  $m \geq 2$ , let the Laurent polynomial  $A$  be defined by (1.3) and (3.1), with the generating function  $Q$  chosen as the symmetric fundamental interpolant of Theorem 3.9. Then, with  $\rho \in \mathbb{N}_m$  denoting the order of zero at  $z = -1$  of the polynomial  $B$ , we have that  $A$  belongs to the class  $\mathcal{A}_{m-1-\nu,\nu}$ , where*

$$\nu = \begin{cases} \frac{1}{2}\rho, & m \text{ even}, \\ \frac{1}{2}(\rho + 1), & m \text{ odd}. \end{cases} \quad (3.52)$$

Moreover,  $A$  satisfies the conditions of Theorem 2.2, with  $n = m - 1$ .

[Recall, as observed before Proposition 3.4, that  $m$  and  $\rho$  are either both even or both odd.]

**Proof.** First, observe from (3.1) and (3.33) that

$$a_j = Q\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad (3.53)$$

and thus, using also (1.3), (3.31) and (3.29), we get, for  $z \in \mathbb{C} \setminus \{0\}$ , that

$$\begin{aligned}
A(z) &= \sum_j Q\left(\frac{j}{2}\right) z^j = \sum_j \sum_k c_k \varphi(j-k) z^j \\
&= z \sum_j \sum_k c_k \varphi(j-k) z^{j-k-1} z^k \\
&= z \sum_j \varphi(j) z^{j-1} \sum_k c_k z^k = z E(z) C(z), \tag{3.54}
\end{aligned}$$

which together with (3.36), yields the formula

$$A(z) = z^{-2m+3} B(z) E(z) G(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.55}$$

with the polynomial  $G$  as in the proof of Theorem 3.7. It then follows from the hypothesis that the polynomial  $B$  has a zero of order  $\rho$  at  $z = -1$ , together with the result (3.43) of Proposition 3.8, that condition (2.42) is satisfied with  $\nu$  given by (3.52), and thus, from Proposition 2.5, that the polynomial filling property (2.38) holds with  $\nu$  given by (3.52).

Next, we observe from (3.55), since also Proposition 3.6 gives  $\deg(E) = m - 2$ , and, since the proof of Theorem 3.7 gives  $\deg(G) = 2m - 4$ , that we have  $\deg(A) = 2m - 3$ . Also, from Propositions 3.8 and 3.4, we know that  $E$  and  $G$  are both symmetric polynomials, from which, together with the symmetry of the polynomial  $B$ , and the formula (3.55), we deduce that  $A$  is a symmetric Laurent polynomial, i.e. (2.36) holds.

Noting from (3.1), (3.33) and Proposition 2.1 that (2.37) holds, we conclude that  $A$  satisfies the conditions of Definition 2.4 with  $\mu = m - 1 - \nu$ , and with  $\nu$  given by (3.52). It follows that  $A \in \mathcal{A}_{m-1-\nu, \nu}$ .

But then, using also (2.41), we see that the conditions (2.21) and (2.22) of Theorem 2.2 hold. Also, since the polynomial  $B$  is symmetric and has only negative zeros, and thus, according to Proposition 3.6 and 3.8, the polynomial  $E$  is symmetric and has only negative zeros, we have that the polynomial  $H$  as given by (3.35) is symmetric and has only negative zeros. Thus, from Proposition 3.4, (3.55), and the fact that  $A(1) = 2$ , from (2.41), we conclude that the positivity condition (2.23) is satisfied.

Finally, since  $\deg(A) = 2m - 3$ , as noted above, we conclude that  $A$  does indeed satisfy the conditions of Theorem 2.2 with  $n = m - 1$ .  $\blacksquare$

**Remarks:**

- (a) Observe from (3.52) that, if  $m$  is an odd integer  $\geq 3$ , then the polynomial reproduction property (2.54) of the corresponding refinable function  $\phi$  holds for  $\nu = \frac{1}{2}(\rho + 1)$ , i.e.

$$\sum_j p(j)\phi(\cdot - j) = p, \quad p \in \pi_\rho, \quad (3.56)$$

whereas the generating function  $Q$  has, according to (3.34) in Theorem 3.7, the same polynomial reproduction property only for polynomials of degree  $\leq (\rho - 1)$ . Hence we did not appeal directly to Theorem 3.1 for the proof of Theorem 3.10, since that result would not have yielded the improved polynomial reproduction degree  $\rho$  in (3.56).

- (b) Observing from (3.52), together with the fact that the order  $\rho$  of the zero at  $z = -1$  of the polynomial  $B$  satisfies  $\rho \in \mathbb{N}_m$ , we see that  $A$  belongs to the class  $\mathcal{A}_{m-1-\nu,\nu}$ , where

$$\nu \in \begin{cases} \mathbb{N}_{\frac{m}{2}}, & m \text{ even}, \\ \mathbb{N}_{\frac{m+1}{2}}, & m \text{ odd}. \end{cases} \quad (3.57)$$

We deduce from (3.57) that  $m - 1 - \nu = 0$  if and only if

$$m \in \{2, 3\} \quad \text{and} \quad \nu = \begin{cases} 1, & m = 2, \\ 2, & m = 3. \end{cases} \quad (3.58)$$

Hence, we have  $A \in \mathcal{A}_{0,\nu}$  if and only if  $m$  and  $\nu$  are such that (3.58) holds, and thus, according to (2.39), the Laurent polynomial  $A$  of Theorem 3.10 satisfies  $A = D_\nu$ , the Dubuc–Deslauriers mask symbol as given by (2.13) and (2.12), if and only if (3.58) holds.

As a special case, we can choose  $B(z) = 2 \left(\frac{1+z}{2}\right)^m$ ,  $z \in \mathbb{C}$ , in Theorem 3.9, i.e.  $Q = Q_m$  is the minimally supported fundamental spline interpolant (3.49) with knots at the half

integers. According to the formula (3.54) we then have  $A(z) = z C_m(z) E_m(z)$ ,  $z \in \mathbb{C} \setminus \{0\}$ .

Using (3.51), together with (3.48) and (3.50), we find that, for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$A(z) = \frac{1}{2} (z^{-1} + 2 + z), \quad m = 2, \quad (3.59)$$

$$A(z) = \frac{1}{16} (-z^{-3} + 9z^{-1} + 16 + 9z - z^3), \quad m = 3, \quad (3.60)$$

$$A(z) = \frac{1}{288} (z^{-5} - 21z^{-3} + 164z^{-1} + 288 + 164z - 21z^3 + z^5), \quad m = 4. \quad (3.61)$$

Note in particular from (3.59), (2.17), (3.60) and (2.18), that  $A = D_1$  and  $A = D_2$ , which is in accordance with our Remark (b) above.

In Figures 3.6 and 3.7, we have plotted the refinable function  $\phi$  associated with mask symbol (3.61), and illustrated the associated convergent subdivision scheme  $S_a$ .

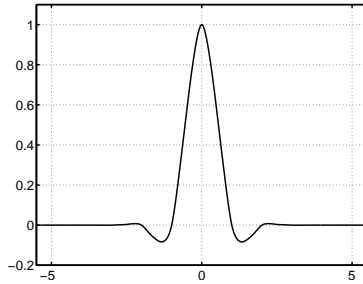


Figure 3.6: Associated refinable function  $\phi$  for the mask symbol (3.61).

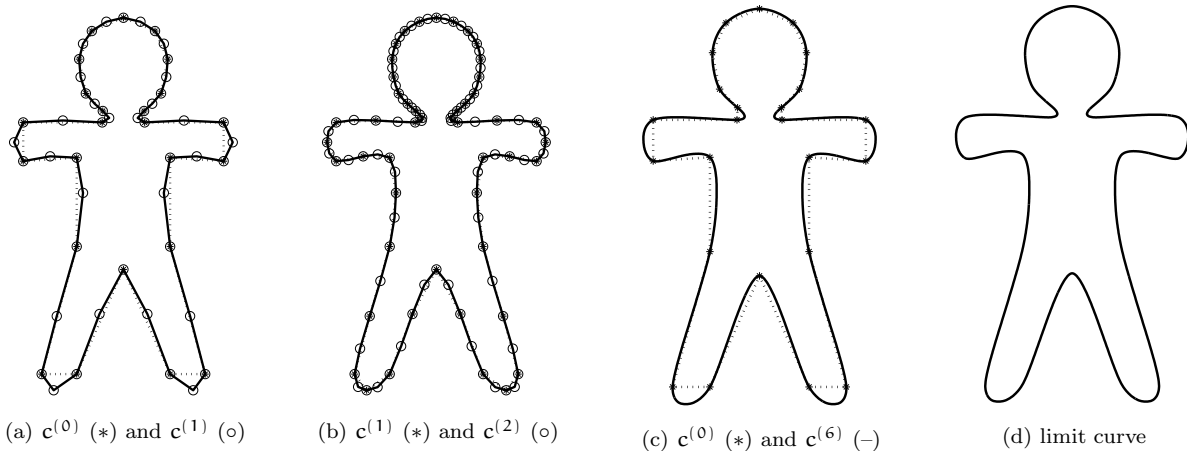


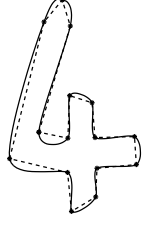
Figure 3.7: Subdivision with the mask symbol (3.61).



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# Existence and convergence by means of the cascade algorithm

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In Chapter 2, we observed that, if a mask symbol  $A$  belongs to the class  $\mathcal{A}_{\mu, \nu}$  for some  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ , then the interpolatory condition (2.21) and condition (2.22) of a zero at  $z = -1$  of Theorem 2.2 are automatically satisfied. According to Theorem 2.2, the further constraint (2.23) of positivity on the unit circle in  $\mathbb{C}$  then guarantees the existence of a corresponding refinable function  $\phi$ , as well as the convergence of the associated subdivision scheme  $S_a$ .

In Section 4.1 below, we provide, as was done in [22], an example of a one-parameter family of mask symbols  $A \in \mathcal{A}_{\mu, \nu}$  that do not satisfy (2.23), but for which numerical evidence seems to suggest the existence of a refinable function, as well as subdivision convergence. Motivated by these results from [22], we proceed here, in Section 4.2, to develop alternative sufficient conditions on mask symbols in  $\mathcal{A}_{\mu, \nu}$  in order to capture existence and convergence for a class of mask symbols which could also include mask symbols not satisfying the positivity condition (2.23), and we then proceed in Section 4.3 to show that a subclass of our example in Section 4.1 does indeed satisfy these conditions.

## 4.1 A family of mask symbols with zeros on the unit circle in $\mathbb{C}$

In the setting of Theorem 2.7, we first show that, for an integer  $n \geq 2$ , the one-parameter family of mask symbols  $A \in \mathcal{A}_{1, n-1}$  given by

$$A = A(t|\cdot) = (1 - t)D_{n-1} + tD_n, \quad t \in \mathbb{R}, \quad (4.1)$$

satisfies  $A(e^{\pm i x_t}) = 0$  for some  $x_t \in (\frac{\pi}{2}, \pi)$  if  $t > 1$ . Hence, this particular one-parameter family of mask symbols  $A$  do not satisfy the positivity condition (2.23) of Theorem 2.2.

We shall rely on the following properties of the Dubuc–Deslauriers mask symbol  $D_n$ .

LEMMA 4.1 *For  $n \in \mathbb{N}$ , the Dubuc–Deslauriers mask symbol  $D_n$ , as given by (2.13) and (2.12), can be generated recursively on the unit circle  $z = e^{ix}$ ,  $x \in \mathbb{R}$ , by using the formulas*

$$D_1(e^{ix}) = \cos x + 1, \quad x \in \mathbb{R}, \quad (4.2)$$

and

$$D_n(e^{ix}) = D_{n-1}(e^{ix}) + \frac{1}{2^{2n-2}} \binom{2n-2}{n-1} \cos x (\sin x)^{2n-2}, \quad x \in \mathbb{R}, \quad n \geq 2. \quad (4.3)$$

Moreover, for all  $n \in \mathbb{N}$ , we have that

$$D_n(1) = 2, \quad D_n(i) = 1, \quad D_n(-1) = 0. \quad (4.4)$$

**Proof.** Setting  $n = 1$  in (2.35), we get

$$D_1(e^{ix}) = \int_x^\pi \sin \omega \, d\omega = \cos x + 1, \quad x \in \mathbb{R},$$

thereby proving (4.2).

Next, for  $n \geq 2$ , we use (2.35) and integration by parts to obtain, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} D_n(e^{ix}) &= \frac{(2n-1)!}{2^{2n-2}[(n-1)!]^2} \int_x^\pi (\sin \omega)^{2n-1} \, d\omega \\ &= \frac{(2n-1)!}{2^{2n-2}[(n-1)!]^2} \left[ \frac{1}{2n-1} \cos x (\sin x)^{2n-2} + \frac{2n-2}{2n-1} \int_x^\pi (\sin \omega)^{2n-3} \, d\omega \right] \\ &= \frac{(2n-2)!}{2^{2n-2}[(n-1)!]^2} \cos x (\sin x)^{2n-2} + \frac{(2n-3)!}{2^{2n-4}[(n-2)!]^2} \int_x^\pi (\sin \omega)^{2n-3} \, d\omega \\ &= \frac{1}{2^{2n-2}} \binom{2n-2}{n-1} \cos x (\sin x)^{2n-2} + D_{n-1}(e^{ix}), \end{aligned}$$

thereby showing that (4.3) holds.

Next, since  $D_n \in \mathcal{A}_{0,n}$ ,  $n \in \mathbb{N}$ , we have from (2.36) that  $D_n(i) = D_n(-i)$ ,  $n \in \mathbb{N}$ , and from (2.37) that  $D_n(i) + D_n(-i) = 2$ , from which we then deduce that  $D_n(i) = 1$ . As

noted before in (2.41), the remaining two values of  $D_n$  in (4.4) are true for all mask symbols in  $\mathcal{A}_{\mu, \nu}$ , and are therefore also true for  $D_n$ .  $\blacksquare$

The following result from [22] provides a class of mask symbols  $\mathcal{A}_{1, n-1}$  that do not satisfy the positivity condition (2.23).

**PROPOSITION 4.2** *For a integer  $n \geq 2$ , the mask symbol  $A(t|\cdot)$  defined by (4.1) satisfies*

$$A(t|e^{ix}) = D_{n-1}(e^{ix}) + t \frac{1}{2^{2n-2}} \binom{2n-2}{n-1} \cos x (\sin x)^{2n-2}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (4.5)$$

with  $D_n$  denoting the Dubuc–Deslauriers mask symbol of order  $n$ . Moreover,

$$\left. \begin{aligned} A(t|e^{i\pi}) &= 0, \\ A(t|e^{ix}) &> 0, \quad x \in (-\pi, \pi), \end{aligned} \right\} \quad t \in [-2n+2, 1], \quad (4.6)$$

whereas, if  $t > 1$ , then there exists exactly one point  $x_t \in (\frac{\pi}{2}, \pi)$  such that

$$\left. \begin{aligned} A(t|e^{ix}) &> 0, \quad x \in (-x_t, x_t) \\ A(t|e^{\pm ix_t}) &= 0, \\ A(t|e^{ix}) &< 0, \quad x \in (-\pi, \pi) \setminus [-x_t, x_t], \\ A(t|e^{i\pi}) &= 0. \end{aligned} \right\} \quad (4.7)$$

**Proof.** To prove (4.5), we first rewrite (4.1) as

$$A(t|z) = D_{n-1}(z) + t \left( D_n(z) - D_{n-1}(z) \right), \quad z \in \mathbb{C} \setminus \{0\},$$

and then use (4.3).

For  $0 \leq t \leq 1$ , (4.6) follows from (4.1) and (2.35). Suppose therefore that  $t \in \mathbb{R} \setminus [0, 1]$ .

From (4.4) and (4.1) we have

$$A(t|1) = 2, \quad A(t|i) = 1, \quad A(t|-1) = 0, \quad t \in \mathbb{R}. \quad (4.8)$$

Now use (2.35) and (4.5) to obtain

$$\frac{d}{dx} A(t|e^{ix}) = \frac{2n-1}{2^{2n-3}} \binom{2n-3}{n-1} (\sin x)^{2n-3} t \left[ \cos^2 x - \frac{1 + \frac{2(n-1)}{t}}{2n-1} \right], \quad x \in \mathbb{R}. \quad (4.9)$$

In (4.9), we find that

$$0 < \frac{1 + \frac{2(n-1)}{t}}{2n-1} < 1, \quad t \in \mathbb{R} \setminus [-2n+2, 1], \quad (4.10)$$

whereas

$$\frac{1 + \frac{2(n-1)}{t}}{2n-1} < 0, \quad t \in (-2n+2, 0). \quad (4.11)$$

Suppose now that  $t \in (-2n+2, 0)$ . Then (4.9) and (4.11) imply that  $\frac{d}{dx}A(t|e^{ix}) < 0$ ,  $x \in [0, \pi)$ , which, together with (4.8), gives  $A(t|x) > 0$ ,  $x \in [0, \pi)$ . But, the fact that  $A \in \mathcal{A}_{1,n-1}$  implies that  $A$  is a symmetric Laurent polynomial, so that

$$A(t|e^{-ix}) = A(t|e^{ix}), \quad x \in \mathbb{R}. \quad (4.12)$$

Hence, if  $t \in (-2n+2, 0)$ , we also have  $A(t|e^{ix}) > 0$ ,  $x \in (-\pi, 0]$ . Since also  $A(t|-1) = 0$  from (4.8), it follows that (4.6) holds for  $t \in (-2n+2, 0)$ .

If  $t = -2n+2$ , we see from (4.9) that

$$\frac{d}{dx}A(t|e^{ix}) = -\frac{(2n-1)(n-1)}{2^{2n-4}} \binom{2n-3}{n-1} (\sin x)^{2n-3} \cos^2 x, \quad x \in \mathbb{R},$$

which, together with (4.8), and the fact that  $A$  is symmetric, completes our proof of (4.6).

Next, suppose  $t \in (1, \infty)$ . Then (4.10) and (4.9) imply the limits

$$\frac{d}{dx}A(t|e^{ix}) \longrightarrow 0^+, \quad x \longrightarrow 0^+,$$

and

$$\frac{d}{dx}A(t|e^{ix}) \longrightarrow 0^+, \quad x \longrightarrow \pi^-.$$

Moreover, from (4.9) and (4.10), there exist precisely two solutions  $x_0 \in (0, \frac{\pi}{2})$  and  $x_1 \in (\frac{\pi}{2}, \pi)$  of the equation  $\frac{d}{dx}A(t|e^{ix}) = 0$  for  $x \in (0, \pi)$ . Keeping the symmetry property (4.12) of  $A$  in mind, as well as (4.8), we deduce that there exists exactly one point  $x_t \in (\frac{\pi}{2}, \pi)$  for which (4.7) holds. ■

Proposition 4.2 shows that if  $t > 1$ , the mask symbol  $A(t|\cdot)$  defined by (4.1) does not satisfy the positivity condition (2.23), so that Theorem 2.2 is inconclusive regarding the

existence of a corresponding refinable function, or the convergence of the associated subdivision scheme  $S_a$ .

However, setting  $n = 2$  and  $t = \frac{3}{2}$  in (4.1), we obtain, using also (2.17) and (2.18), the formula

$$A\left(\frac{3}{2}|z\right) = -\frac{3}{32}z^{-3} + \frac{19}{32}z^{-1} + 1 + \frac{19}{32}z - \frac{3}{32}z^3, \quad z \in \mathbb{C} \setminus \{0\}; \quad (4.13)$$

and if we now apply the associated subdivision scheme  $S_a$  to the initial sequence  $c = \delta$ , our numerical calculations, as illustrated in Figure 4.1, seem to suggest convergence, to what would then be the corresponding refinable function  $\phi$ .

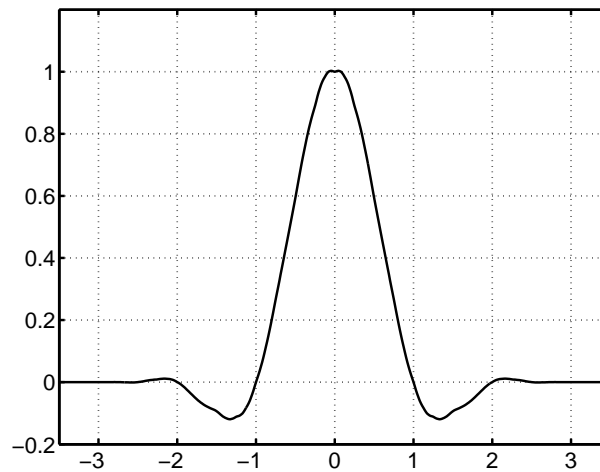


Figure 4.1: Numerical evidence of convergence of  $S_a$  with mask symbol (4.13)

In Section 4.2 below, we proceed to develop sufficient conditions for existence and convergence that will also capture a collection of mask symbols for which Theorem 2.2 is not valid due to the non-compliance of the mask symbol to the positivity condition (2.23).

## 4.2 A convergence theorem

In this section we derive, in the setting of Theorem 2.2, an alternative to the positivity condition (2.23), while still ensuring the existence of an interpolatory refinable function and the convergence of the associated interpolatory subdivision scheme.

### 4.2.1 The cascade algorithm

We define, for a given mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , the ***cascade operator***  $T_{\mathbf{a}} : M(\mathbb{R}) \rightarrow M(\mathbb{R})$  by

$$(T_{\mathbf{a}}f)(x) = \sum_j \mathbf{a}_j f(2x - j), \quad x \in \mathbb{R}. \quad (4.14)$$

The corresponding ***cascade algorithm***, then generates, for a given initial function  $g \in M(\mathbb{R})$ , the sequence  $\{\phi_r : r \in \mathbb{Z}_+\} \subset M(\mathbb{R})$  recursively by

$$\phi_0 = g, \quad \phi_{r+1} = T_{\mathbf{a}}\phi_r, \quad r \in \mathbb{Z}_+. \quad (4.15)$$

We shall rely on the following relationship, (see [50],[20]), between the subdivision and cascade operators.

**LEMMA 4.3** *For a given mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , the subdivision operator  $S_{\mathbf{a}} : M(\mathbb{Z}) \rightarrow M(\mathbb{Z})$  and the cascade operator  $T_{\mathbf{a}} : M(\mathbb{R}) \rightarrow M(\mathbb{R})$ , as defined by, respectively, (1.2) and (4.14), satisfy the relationship*

$$(T_{\mathbf{a}}^r f)(x) = \sum_j (S_{\mathbf{a}}^r \delta)_j f(2^r x - j), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}, \quad f \in M(\mathbb{R}), \quad (4.16)$$

with  $\delta = \{\delta_j : j \in \mathbb{Z}\}$ .

**Proof.** Since  $S_{\mathbf{a}}\delta = \mathbf{a}$  from (1.2) and (1.27), we see from (4.14) that (4.16) holds for  $r = 1$ . If  $r \geq 2$ , we use (1.2) and (4.14) to get, for  $f \in M(\mathbb{R})$  and  $x \in \mathbb{R}$ ,

$$\begin{aligned} \sum_j (S_{\mathbf{a}}^r \delta)_j f(2^r x - j) &= \sum_j \left[ \sum_k \mathbf{a}_{j-2k} (S_{\mathbf{a}}^{r-1} \delta)_k \right] f(2^r x - j) \\ &= \sum_k (S_{\mathbf{a}}^{r-1} \delta)_k \left[ \sum_j \mathbf{a}_{j-2k} f(2^r x - j) \right] \\ &= \sum_k (S_{\mathbf{a}}^{r-1} \delta)_k \left[ \sum_j \mathbf{a}_j f(2(2^{r-1}x - k) - j) \right] \\ &= \sum_k (S_{\mathbf{a}}^{r-1} \delta)_k (T_{\mathbf{a}}f)(2^{r-1}x - k) \\ &\quad \vdots \\ &= \sum_k \mathbf{a}_k (T_{\mathbf{a}}^{r-1}f)(2x - k) = (T_{\mathbf{a}}^r f)(x). \quad \blacksquare \end{aligned}$$

## 4.2.2 Sufficient conditions for existence and convergence

First, for a given interpolatory mask  $\mathbf{a} \in M_0(\mathbb{Z})$ , we establish, in terms of the associated interpolatory subdivision scheme  $S_{\mathbf{a}}$ , a sufficient condition for the convergence of the cascade algorithm (4.15) for a suitable choice of initial function  $g$ .

We denote by  $C_u(\mathbb{R})$  the space of bounded functions in  $C(\mathbb{R})$ , i.e.  $f \in C_u(\mathbb{R})$  if  $f \in C(\mathbb{R})$  and  $\|f\|_{\infty} := \sup_{x \in \mathbb{R}} |f(x)| < \infty$ , where we define  $\sup_x = \sup_{x \in \mathbb{R}}$ . Recall that  $C_u(\mathbb{R})$  is a complete normed linear space (or Banach space) with respect to the norm  $\|\cdot\|_{\infty}$ .

Note that we use the same notation  $\|\cdot\|_{\infty}$  for the norm on  $\ell^{\infty}(\mathbb{Z})$  and the norm on  $C_u(\mathbb{R})$ ; however the appropriate meaning should always be clear from the context.

**THEOREM 4.4** *Suppose the mask  $\mathbf{a} \in M_0(\mathbb{Z})$  also satisfies*

$$a_{2j} = \delta_j, \quad j \in \mathbb{Z}, \quad (4.17)$$

$$\sum_j a_{2j+1} = 1, \quad (4.18)$$

and let  $\{\phi_r : r \in \mathbb{Z}_+\}$  denote the sequence in  $C(\mathbb{R})$  generated by the cascade algorithm (4.15) with the initial function  $g$  chosen as

$$g = N_2(\cdot + 1), \quad (4.19)$$

where  $N_2$  is the linear B-spline, as given by (1.23).

Also, with the choice  $\mathbf{c} = \delta$  in (1.4), let, according to Proposition 1.1,  $\{\mathbf{c}^{(r)} : r \in \mathbb{Z}_+\} \subset \ell^{\infty}(\mathbb{Z})$  denote the sequence generated, according to (1.5), by the subdivision scheme  $S_{\mathbf{a}}$ , and define the sequence  $\{\gamma^{(r)} : r \in \mathbb{Z}_+\} \subset \ell^{\infty}(\mathbb{Z})$  by

$$\gamma_j^{(r)} = c_j^{(r)} - \frac{1}{2} \left( c_{j-1}^{(r)} + c_{j+1}^{(r)} \right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (4.20)$$

If, moreover, there exists a real number  $\rho = \rho(\mathbf{a}) \in [0, 1)$  such that

$$\|\gamma^{(r+1)}\|_{\infty} \leq \rho \|\gamma^{(r)}\|_{\infty}, \quad r \in \mathbb{Z}_+, \quad (4.21)$$

then the cascade algorithm (4.15) converges in the sense that there exists a function  $\phi \in C_u(\mathbb{R})$  such that

$$\|\phi - \phi_r\|_\infty \leq \frac{\rho^{r+1}}{1-\rho} \longrightarrow 0, \quad r \longrightarrow \infty. \quad (4.22)$$

**Proof.** First, observe from (4.19), (1.23), (1.15) and (1.16) that  $g \in C_0(\mathbb{R})$ , and that

$$g = \sum_j b_j g(2 \cdot -j), \quad \text{where} \quad b_j = \begin{cases} \frac{1}{2}, & j = \pm 1, \\ 1, & j = 0, \\ 0, & j \notin \{-1, 0, 1\}. \end{cases} \quad (4.23)$$

Now use Lemma 4.3, together with (1.5), (4.17), Proposition 2.1, (2.2), (4.23) (1.18) and (1.19), to deduce that, for  $r \in \mathbb{Z}_+$  and  $x \in \mathbb{R}$ , we have

$$\begin{aligned} |\phi_{r+1}(x) - \phi_r(x)| &= \left| \sum_j (S_a^{r+1} \delta)_j g(2^{r+1}x - j) - \sum_j (S_a^r \delta)_j g(2^r x - j) \right| \\ &= \left| \sum_j c_j^{(r+1)} g(2^{r+1}x - j) - \sum_j c_j^{(r)} g(2^r x - j) \right| \\ &= \left| \sum_j c_j^{(r+1)} g(2^{r+1}x - j) - \sum_j c_{2j}^{(r+1)} \sum_k b_k g(2^{r+1}x - 2j - k) \right| \\ &= \left| \sum_j c_j^{(r+1)} g(2^{r+1}x - j) - \sum_j c_{2j}^{(r+1)} \sum_k b_{k-2j} g(2^{r+1}x - k) \right| \\ &= \left| \sum_j c_j^{(r+1)} g(2^{r+1}x - j) - \sum_j \left( \sum_k b_{j-2k} c_{2k}^{(r+1)} \right) g(2^{r+1}x - j) \right| \\ &= \left| \sum_j \left[ c_j^{(r+1)} - \sum_k b_{j-2k} c_{2k}^{(r+1)} \right] g(2^{r+1}x - j) \right| \\ &= \left| \sum_j \left[ c_{2j}^{(r+1)} - \sum_k b_{2j-2k} c_{2k}^{(r+1)} \right] g(2^{r+1}x - 2j) \right. \\ &\quad \left. + \sum_j \left[ c_{2j+1}^{(r+1)} - \sum_k b_{2j+1-2k} c_{2k}^{(r+1)} \right] g(2^{r+1}x - 2j - 1) \right| \\ &= \left| \sum_j \left[ c_{2j+1}^{(r+1)} - \sum_k b_{2j+1-2k} c_{2k}^{(r+1)} \right] g(2^{r+1}x - 2j - 1) \right| \\ &= \left| \sum_j \left[ c_{2j+1}^{(r+1)} - \frac{1}{2} c_{2j}^{(r+1)} - \frac{1}{2} c_{2j+2}^{(r+1)} \right] g(2^{r+1}x - 2j - 1) \right| \end{aligned}$$



$$\begin{aligned}
 &= \left| \sum_j \gamma_{2j+1}^{(r+1)} g(2^{r+1}x - 2j - 1) \right| \\
 &\leq \sum_j \left| \gamma_{2j+1}^{(r+1)} \right| g(2^{r+1}x - 2j - 1) \\
 &\leq \|\gamma^{(r+1)}\|_\infty \sum_j g(2^{r+1}x - 2j - 1) \\
 &\leq \|\gamma^{(r+1)}\|_\infty \sum_j g(2^{r+1}x - j) = \|\gamma^{(r+1)}\|_\infty.
 \end{aligned} \tag{4.24}$$

But (4.21) implies that, for  $r \in \mathbb{Z}_+$ ,

$$\|\gamma^{(r+1)}\|_\infty \leq \rho \|\gamma^{(r)}\|_\infty \leq \rho^2 \|\gamma^{(r-1)}\|_\infty \leq \cdots \leq \rho^{r+1} \|\gamma^{(0)}\|_\infty \leq \rho^{r+1}, \tag{4.25}$$

since  $\|\gamma^{(0)}\|_\infty = 1$ , from the fact that (4.20) yields  $\gamma_j^{(0)} = \delta_j - \frac{1}{2}(\delta_{j-1} + \delta_{j+1})$ ,  $j \in \mathbb{Z}$ . It follows from (4.24) and (4.25) that

$$|\phi_{r+1}(x) - \phi_r(x)| \leq \rho^{r+1}, \quad x \in \mathbb{R}, \quad r \in \mathbb{Z}_+,$$

from which we then deduce that

$$|\phi_{r+\ell}(x) - \phi_r(x)| \leq \frac{\rho^{r+1}}{1-\rho}, \quad x \in \mathbb{R}, \quad r \in \mathbb{Z}_+, \quad \ell \in \mathbb{N}. \tag{4.26}$$

Since (4.14), together with the fact that  $\mathbf{a} \in M_0(\mathbb{Z})$ , implies that the operator  $T_{\mathbf{a}}$  maps  $C_0(\mathbb{R})$  into  $C_0(\mathbb{R}) \subset C_u(\mathbb{R})$ , and since  $g \in C_0(\mathbb{R}) \subset C_u(\mathbb{R})$ , we see from (4.15) that  $\{\phi_r : r \in \mathbb{Z}_+\} \subset C_0(\mathbb{R}) \subset C_u(\mathbb{R})$ . Hence (4.26) yields the estimate

$$\|\phi_{r+\ell} - \phi_r\|_\infty \leq \frac{\rho^{r+1}}{1-\rho}, \quad r \in \mathbb{Z}_+, \quad \ell \in \mathbb{N},$$

so that  $\|\phi_j - \phi_k\|_\infty \rightarrow 0$ ,  $j, k \rightarrow \infty$ , and we conclude that  $\{\phi_r : r \in \mathbb{Z}_+\}$  is a Cauchy sequence in  $C_u(\mathbb{R})$  with respect to the  $\|\cdot\|_\infty$  norm. But  $C_u(\mathbb{R})$  is a complete normed linear space (or Banach space) with respect to the norm  $\|\cdot\|_\infty$ . Hence there exists a limit function  $\phi \in C_u(\mathbb{R})$  such that  $\|\phi - \phi_r\|_\infty \rightarrow 0$ ,  $r \rightarrow \infty$ . Moreover, since, for a fixed  $r \in \mathbb{Z}_+$ , and any  $\ell \in \mathbb{N}$ , we have from (4.26) that

$$\begin{aligned}
 \|\phi - \phi_r\|_\infty &\leq \|\phi - \phi_{r+\ell}\|_\infty + \|\phi_{r+\ell} - \phi_r\|_\infty \\
 &\leq \|\phi - \phi_{r+\ell}\|_\infty + \frac{\rho^r}{1-\rho} \\
 &\rightarrow 0 + \frac{\rho^r}{1-\rho}, \quad \ell \rightarrow \infty.
 \end{aligned}$$

It then follows that (4.22) holds. ■

Next, we prove the following properties of the limit function  $\phi$  in Theorem 4.4.

**THEOREM 4.5** *Suppose, in Theorem 4.4, we also have that*

$$\text{supp}(\mathbf{a}) = [M, \dots, N], \quad (4.27)$$

where  $M, N \in \mathbb{Z}$  satisfy  $M \leq -1$  and  $N \geq 1$ . Then the limit function  $\phi$  in Theorem 4.4 is in  $C_0(\mathbb{R})$ , and satisfies the conditions

$$\phi(x) = 0, \quad x \notin (M, N); \quad (4.28)$$

$$\phi = \sum_j \mathbf{a}_j \phi(2 \cdot -j); \quad (4.29)$$

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}; \quad (4.30)$$

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}. \quad (4.31)$$

Moreover,  $\phi$  is the unique function in  $C_0(\mathbb{R})$  satisfying (4.29) and (4.30).

**Proof.** Our first step is to prove that

$$\phi_r(x) = 0, \quad x \notin \left(M - \frac{M+1}{2^r}, N - \frac{N-1}{2^r}\right), \quad r \in \mathbb{Z}_+, \quad (4.32)$$

which, together with the conditions  $M \leq -1$  and  $N \geq 1$ , would then imply that

$$\phi_r(x) = 0, \quad x \notin (M, N), \quad r \in \mathbb{Z}_+. \quad (4.33)$$

To this end, we first note from (4.19) and (1.23) that

$$g(x) = \begin{cases} x + 1, & x \in (-1, 0), \\ 1 - x, & x \in [0, 1), \\ 0, & x \notin (-1, 1), \end{cases} \quad (4.34)$$

and thus, since  $\phi_0(x) = g(x) = 0$ ,  $x \notin (-1, 1)$ , we see that (4.32) holds for  $r = 0$ .

Proceeding inductively, we next assume that (4.32) holds for a fixed  $r \in \mathbb{Z}_+$ . Then (4.15),

(4.14) and (4.27) imply that (4.32) holds for  $r + 1$ . The general validity of (4.32) follows by induction.

Now let  $x \notin (M, N)$ . Then, (4.33) and (4.22) yield

$$|\phi(x)| = |\phi(x) - \phi_r(x)| \leq \|\phi - \phi_r\|_\infty \longrightarrow 0, \quad r \longrightarrow \infty,$$

thereby proving that (4.28) holds, and thus also that  $\phi \in C_0(\mathbb{R})$ .

To prove the refinability (4.29) of  $\phi$ , we first show that  $T_a$  maps  $C_u(\mathbb{R})$  into itself. To this end, we use (4.27) and (4.14) to obtain, for  $f \in C_u(\mathbb{R})$  and  $x \in \mathbb{R}$ , the estimates

$$|(T_a f)(x)| \leq \sum_{j=M}^N |a_j| |f(2x - j)| \leq \|f\|_\infty \sum_{j=M}^N |a_j|,$$

from which we then deduce that  $T_a$  maps  $C_u(\mathbb{R})$  into itself, with

$$\|T_a f\|_\infty \leq \|f\|_\infty \sum_{j=M}^N |a_j|, \quad f \in C_u(\mathbb{R}).$$

Hence,

$$\frac{\|T_a f\|_\infty}{\|f\|_\infty} \leq \sum_{j=M}^N |a_j|, \quad f \in C_u(\mathbb{R}), \quad f \neq 0,$$

and it follows that  $T_a$  is in fact a bounded linear operator from  $C_u(\mathbb{R})$  to  $C_u(\mathbb{R})$ , with the operator norm  $\|T_a\|_\infty \leq \sum_{j=M}^N |a_j|$ .

According to a standard result (see e.g. [33, Theorem 4.4.2]),  $T_a$  is therefore also a continuous operator from  $C_u(\mathbb{R})$  into itself. It then follows from (4.22), together with the fact that  $\phi \in C_0(\mathbb{R}) \subset C_u(\mathbb{R})$ , that

$$\phi = \lim_{r \rightarrow \infty} \phi_{r+1} = \lim_{r \rightarrow \infty} (T_a \phi_r) = T_a \left( \lim_{r \rightarrow \infty} \phi_r \right) = T_a \phi,$$

and thus  $\phi = T_a \phi$ , which, according to the definition (4.14) of the cascade operator  $T_a$ , is equivalent to the refinement equation (4.29).

Next, to prove (4.30), we first show that

$$\phi_r(j) = \delta_j, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (4.35)$$

Observing from (4.15) and (4.34) that (4.35) holds for  $r = 0$ , we suppose next that (4.35) is true for a fixed  $r \in \mathbb{Z}_+$ . Then (4.14) and (4.15) give, for  $j \in \mathbb{Z}$ ,

$$\phi_{r+1}(j) = \sum_k a_k \phi_r(2j - k) = \sum_k a_{2j-k} \phi_r(k) = \sum_k a_{2j-k} \delta_k = a_{2j} = \delta_j,$$

from (4.17), thereby concluding our inductive proof of (4.35). But then (4.22) and (4.35) yield, for  $j \in \mathbb{Z}$ ,

$$|\phi(j) - \delta_j| = |\phi(j) - \phi_r(j)| \leq \|\phi - \phi_r\|_\infty \longrightarrow 0, \quad r \longrightarrow \infty,$$

from which we deduce that (4.30) is true.

Similarly, we prove (4.31) by first showing inductively that

$$\sum_j \phi_r(x - j) = 1, \quad x \in \mathbb{R}, \quad r \in \mathbb{Z}_+. \quad (4.36)$$

To this end, first note from (4.15), (4.19), and (4.34) that (4.36) is true for  $r = 0$ . We now assume that (4.36) holds for a fixed  $r \in \mathbb{Z}_+$ . But then, from (4.14) and (4.15), and since (4.17) and (4.18) hold, we get, for  $x \in \mathbb{R}$ , that

$$\begin{aligned} \sum_j \phi_{r+1}(x - j) &= \sum_j \left[ \sum_k a_k \phi_r(2x - 2j - k) \right] \\ &= \sum_j \left[ \sum_k a_{k-2j} \phi_r(2x - k) \right] \\ &= \sum_k \left[ \sum_j a_{k-2j} \right] \phi_r(2x - k) \\ &= \sum_k \phi_r(2x - k) = 1, \end{aligned}$$

from our induction hypothesis. Hence (4.36) is true.

Now for a fixed  $x \in \mathbb{R}$ , and suppose  $k$  is the (unique) integer such that  $x \in [k, k + 1)$ .

Using (4.36) and (4.28), we obtain, for  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 \left| \sum_j \phi(x-j) - 1 \right| &= \left| \sum_j [\phi(x-j) - \phi_r(x-j)] \right| \\
 &= \left| \sum_{j=k+1-N}^{k-M} [\phi(x-j) - \phi_r(x-j)] \right| \\
 &\leq \sum_{j=k+1-N}^{k-M} |\phi(x-j) - \phi_r(x-j)| \\
 &\leq (N-M) \|\phi - \phi_r\|_\infty \longrightarrow 0, \quad r \longrightarrow \infty,
 \end{aligned}$$

by virtue of (4.22), and it follows that (4.31) does indeed hold.

Finally, to prove the uniqueness result of the theorem, suppose that  $\psi \in C_0(\mathbb{R})$  satisfies

$$\psi = \sum_j a_j \psi(2 \cdot -j), \quad (4.37)$$

and

$$\psi(j) = \delta_j, \quad j \in \mathbb{Z}, \quad (4.38)$$

and let  $\theta = \phi - \psi$ . Our proof will be complete if we can show that

$$\theta\left(\frac{k}{2^r}\right) = 0, \quad k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+, \quad (4.39)$$

from which it would then follow that  $\theta(x) = 0$ ,  $x \in \mathbb{R}$ , since the dyadic set  $\{\frac{k}{2^r} : k \in \mathbb{Z}, r \in \mathbb{Z}_+\}$  is dense in  $\mathbb{R}$ , and  $\theta \in C_0(\mathbb{R})$ .

To prove (4.39), we first note from (4.30) and (4.38) that (4.39) holds for  $r = 0$ . Next we use (4.29) and (1.2) to deduce that, for  $k \in \mathbb{Z}$  and  $r \in \mathbb{N}$ , and with  $\delta = \{\delta_j : j \in \mathbb{Z}\}$ , we have

$$\begin{aligned}
 \phi\left(\frac{k}{2^r}\right) &= \sum_j a_j \phi\left(\frac{k}{2^{r-1}} - j\right) \\
 &= \sum_j (S_a \delta)_j \phi\left(\frac{k}{2^{r-1}} - j\right) \\
 &= \sum_j (S_a \delta)_j \sum_\ell a_\ell \phi\left(\frac{k}{2^{r-2}} - 2j - \ell\right) \\
 &= \sum_j (S_a \delta)_j \sum_\ell a_{\ell-2j} \phi\left(\frac{k}{2^{r-2}} - \ell\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\ell} \left[ \sum_j a_{\ell-2j} (S_a \delta)_j \right] \phi \left( \frac{k}{2^{r-2}} - \ell \right) \\
 &= \sum_{\ell} (S_a^2 \delta)_{\ell} \phi \left( \frac{k}{2^{r-2}} - \ell \right) \\
 &\quad \vdots \\
 &= \sum_{\ell} (S_a^r \delta)_{\ell} \phi(k - \ell) = (S_a^r \delta)_k,
 \end{aligned} \tag{4.40}$$

having also used (4.30). Similarly, (4.37) and (4.38) yield

$$\psi \left( \frac{k}{2^r} \right) = (S_a^r \delta)_k, \quad k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{4.41}$$

Together, (4.40) and (4.41) then show that (4.39) also holds for  $r \in \mathbb{N}$ . ■

As an immediate consequence of Theorems 4.4 and 4.5, together with Theorem 2.6, we can now state the following result.

**COROLLARY 4.6** *In Theorem 4.4, suppose the mask  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$ , in addition to satisfying (4.17) and (4.18), is such that the corresponding mask symbol  $\mathbf{A}$ , as defined by (1.3), belongs to the class  $\mathcal{A}_{\mu, \nu}$  for some  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ . Then Theorem 4.5 holds with  $\mathbf{M} = -2\mu - 2\nu + 1$  and  $\mathbf{N} = 2\mu + 2\nu - 1$ , and the corresponding refinable function  $\phi$  satisfies the properties (2.54) – (2.57) of Theorem 2.6.*

Next, regarding the convergence of the subdivision scheme  $S_a$  associated with the mask  $\mathbf{a}$  of Theorem 4.4, we note that the existence of the corresponding refinable function  $\phi$ , together with the properties (4.29) and (4.30) can be used, as in the proof of Theorem 2.2 (see also Remark (b) after that proof), to immediately yield the following result.

**THEOREM 4.7** *The subdivision scheme  $S_a$ , as defined by (1.2) and (1.4), and with mask  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$  as in Theorem 4.4, is interpolatory in the sense of (2.1), and converges on  $\mathbf{M}(\mathbb{Z})$ , in the sense that (2.28) holds, and with the limit function  $\Phi$  given by (2.27).*

To complete our existence and convergence theory of this section, we establish sufficient conditions on the mask  $\mathbf{a}$  of Theorem 4.4 for the contractive property (4.21) to hold.

**THEOREM 4.8** *Suppose the mask symbol  $\mathbf{A}$  associated with the mask  $\mathbf{a} \in \mathbf{M}_0(\mathbb{Z})$  in Theorem 4.4 is such that  $\mathbf{A}$  belongs to the class  $\mathcal{A}_{\mu, \nu}$  for some  $\mu \in \mathbb{Z}_+$  and  $\nu \in \mathbb{N}$ , with  $n = \mu + \nu \geq 2$ . Suppose also, with the sequence  $\alpha = \{\alpha_\ell : \ell \in \mathbb{Z}\} \in \mathbf{M}_0(\mathbb{Z})$  defined by*

$$\alpha_\ell = \begin{cases} \sum_{k=\ell+1}^{n-1} (k-\ell) \mathbf{a}_{2k+1}, & \ell \in \mathbb{Z}_{n-2}, \\ 0, & \ell \notin \mathbb{Z}_{n-2}, \end{cases} \quad (4.42)$$

*that the two inequalities*

$$\sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell| < \frac{1}{4}, \quad (4.43)$$

*and*

$$\sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell| < \frac{1}{4}, \quad (4.44)$$

*are satisfied. Then there exists a real number  $\rho = \rho(\mathbf{a}) \in [0, 1)$  such that the inequality (4.21) holds, so that all the conclusions of Theorems 4.4 and 4.5, Corollary 4.6 and Theorem 4.7 are valid.*

**Proof.** We first prove that the sequence  $\{\gamma^{(r)} : r \in \mathbb{Z}_+\} \subset \ell^\infty(\mathbb{Z})$ , as given by (4.20), satisfies the recursion formulas

$$\gamma_{2j+1}^{(r+1)} = -2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right), \quad j \in \mathbb{Z}, r \in \mathbb{Z}_+, \quad (4.45)$$

$$\gamma_{2j}^{(r+1)} = \frac{1}{2} \left[ \gamma_j^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} (\alpha_{\ell-1} + \alpha_\ell) \left( \gamma_{j+\ell}^{(r)} + \gamma_{j-\ell}^{(r)} \right) \right], \quad j \in \mathbb{Z}, r \in \mathbb{Z}_+. \quad (4.46)$$

To this end, we first use (1.2), (1.4), and (2.53) and (2.52), to obtain, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned} \mathbf{c}_{2j+1}^{(r+1)} &= \sum_{k \in \{j + \mathbb{J}_n\}} \mathbf{a}_{2j+1-2k} \mathbf{c}_k^{(r)} = \sum_{k \in \mathbb{J}_n} \mathbf{a}_{1-2k} \mathbf{c}_{k+j}^{(r)} \\ &= \sum_{k=-n+1}^0 \mathbf{a}_{1-2k} \mathbf{c}_{k+j}^{(r)} + \sum_{k \in \mathbb{N}_n} \mathbf{a}_{1-2k} \mathbf{c}_{k+j}^{(r)} \\ &= \sum_{k \in \mathbb{N}_n} \mathbf{a}_{2k-1} \left( \mathbf{c}_{1-k+j}^{(r)} + \mathbf{c}_{k+j}^{(r)} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{Z}_{n-1}} a_{2k+1} \left( c_{j-k}^{(r)} + c_{j+k+1}^{(r)} \right) \\
 &= \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \left( c_{j-k}^{(r)} + c_{j+k+1}^{(r)} \right) + a_1 \left( c_j^{(r)} + c_{j+1}^{(r)} \right). \tag{4.47}
 \end{aligned}$$

But (2.40), (2.53), (4.18) and (2.52) give

$$1 = \sum_j a_{2j+1} = \sum_{j=-n}^{-1} a_{2j+1} + \sum_{j \in \mathbb{Z}_{n-1}} a_{2j+1} = \sum_{j \in \mathbb{Z}_{n-1}} (a_{-2j-1} + a_{2j+1}) = 2 \sum_{j \in \mathbb{Z}_{n-1}} a_{2j+1},$$

and thus

$$\sum_{j \in \mathbb{Z}_{n-1}} a_{2j+1} = \frac{1}{2}. \tag{4.48}$$

Combining (4.47) and (4.48) then yields, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 c_{2j+1}^{(r+1)} &= \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \left( c_{j-k}^{(r)} + c_{j+k+1}^{(r)} \right) + \left( \frac{1}{2} - \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \right) \left( c_j^{(r)} + c_{j+1}^{(r)} \right) \\
 &= \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \left( c_{j-k}^{(r)} - c_{j+1}^{(r)} - c_j^{(r)} + c_{j+k+1}^{(r)} \right) + \frac{1}{2} \left( c_j^{(r)} + c_{j+1}^{(r)} \right). \tag{4.49}
 \end{aligned}$$

Now we use (4.20), (4.49) and (2.2) to obtain, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 \gamma_{2j+1}^{(r+1)} &= c_{2j+1}^{(r+1)} - \frac{1}{2} \left( c_{2j}^{(r+1)} + c_{2j+2}^{(r+1)} \right) \\
 &= c_{2j+1}^{(r+1)} - \frac{1}{2} \left( c_j^{(r)} + c_{j+1}^{(r)} \right) \\
 &= \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \left( c_{j-k}^{(r)} - c_{j+1}^{(r)} - c_j^{(r)} + c_{j+k+1}^{(r)} \right). \tag{4.50}
 \end{aligned}$$

We claim that, for  $j \in \mathbb{Z}$ ,  $k \in \mathbb{N}$  and  $r \in \mathbb{Z}_+$ ,

$$c_{j-k}^{(r)} - c_{j+1}^{(r)} - c_j^{(r)} + c_{j+k+1}^{(r)} = -2 \sum_{\ell \in \mathbb{Z}_{k-1}} (k - \ell) \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right). \tag{4.51}$$

To prove (4.51), we insert the definition (4.20) into the right-hand side of (4.51) to find that

$$\begin{aligned}
 &-2 \sum_{\ell \in \mathbb{Z}_{k-1}} (k - \ell) \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right) \\
 &= \sum_{\ell \in \mathbb{Z}_{k-1}} (k - \ell) \left( c_{j+\ell+2}^{(r)} - 2c_{j+\ell+1}^{(r)} + c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} - 2c_{j-\ell}^{(r)} + c_{j-\ell-1}^{(r)} \right)
 \end{aligned}$$



$$\begin{aligned}
 &= \sum_{\ell \in \mathbb{Z}_{k-1}} (k-\ell) \left( c_{j+\ell+2}^{(r)} + c_{j-\ell-1}^{(r)} \right) - 2 \sum_{\ell \in \mathbb{Z}_{k-1}} (k-\ell) \left( c_{j+\ell+1}^{(r)} + c_{j-\ell}^{(r)} \right) \\
 &\quad + \sum_{\ell \in \mathbb{Z}_{k-1}} (k-\ell) \left( c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} \right) \\
 &= \sum_{\ell=2}^{k+1} (k-\ell+2) \left( c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} \right) - 2 \sum_{\ell \in \mathbb{N}_k} (k-\ell+1) \left( c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} \right) \\
 &\quad + \sum_{\ell \in \mathbb{Z}_{k-1}} (k-\ell) \left( c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} \right) \\
 &= k \left( c_j^{(r)} + c_{j+1}^{(r)} \right) + (k-1) \left( c_{j+1}^{(r)} + c_j^{(r)} \right) - 2k \left( c_{j+1}^{(r)} + c_j^{(r)} \right) \\
 &\quad + \sum_{\ell=2}^{k-1} [k-\ell+2 - 2(k-\ell+1) + k-\ell] \left( c_{j+\ell}^{(r)} + c_{j-\ell+1}^{(r)} \right) \\
 &\quad + \left( c_{j+k+1}^{(r)} + c_{j-k}^{(r)} \right) - 2 \left( c_{j+k}^{(r)} + c_{j-k+1}^{(r)} \right) + 2 \left( c_{j+k}^{(r)} + c_{j-k+1}^{(r)} \right) \\
 &= - \left( c_{j+1}^{(r)} + c_j^{(r)} \right) + \left( c_{j+k+1}^{(r)} + c_{j-k}^{(r)} \right),
 \end{aligned}$$

which is the left-hand side of (4.51).

Now substitute (4.51) into (4.50) to obtain, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 \gamma_{2j+1}^{(r+1)} &= -2 \sum_{k \in \mathbb{N}_{n-1}} a_{2k+1} \sum_{\ell \in \mathbb{Z}_{k-1}} (k-\ell) \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right) \\
 &= -2 \sum_{\ell \in \mathbb{Z}_{n-2}} \left[ \sum_{k=\ell+1}^{n-1} (k-\ell) a_{2k+1} \right] \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right) = -2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \left( \gamma_{j+\ell+1}^{(r)} + \gamma_{j-\ell}^{(r)} \right),
 \end{aligned}$$

from (4.42), and thereby proving the recursion formula (4.45).

To prove (4.46), we first show that

$$\gamma_{2j}^{(r+1)} = \frac{1}{2} \left( \gamma_j^{(r)} - \gamma_{2j+1}^{(r+1)} - \gamma_{2j-1}^{(r+1)} \right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (4.52)$$

To this end, we first use (4.20) and (2.2) to obtain, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 &\gamma_j^{(r)} - \gamma_{2j+1}^{(r+1)} - \gamma_{2j-1}^{(r+1)} \\
 &= \frac{1}{2} \left[ 2c_j^{(r)} - c_{j-1}^{(r)} - c_{j+1}^{(r)} - 2c_{2j+1}^{(r+1)} + c_{2j}^{(r+1)} + c_{2j+2}^{(r+1)} - 2c_{2j-1}^{(r+1)} + c_{2j-2}^{(r+1)} + c_{2j}^{(r+1)} \right] \\
 &= 2c_{2j}^{(r+1)} - c_{2j+1}^{(r+1)} - c_{2j-1}^{(r+1)} = 2\gamma_{2j}^{(r+1)},
 \end{aligned}$$

thereby proving (4.52). Now we substitute (4.45) into (4.52) to get, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 2\gamma_{2j}^{(r+1)} &= \gamma_j^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j+\ell+1}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j-\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j+\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j-1-\ell}^{(r)} \\
 &= \gamma_j^{(r)} + 2 \sum_{\ell \in \mathbb{N}_{n-1}} \alpha_{\ell-1} \gamma_{j+\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j-\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-2}} \alpha_\ell \gamma_{j+\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{N}_{n-1}} \alpha_{\ell-1} \gamma_{j-\ell}^{(r)} \\
 &= \gamma_j^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} (\alpha_{\ell-1} + \alpha_\ell) \gamma_{j+\ell}^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} (\alpha_{\ell-1} + \alpha_\ell) \gamma_{j-\ell}^{(r)} \\
 &= \gamma_j^{(r)} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} (\alpha_{\ell-1} + \alpha_\ell) (\gamma_{j+\ell}^{(r)} + \gamma_{j-\ell}^{(r)}),
 \end{aligned}$$

after having recalled also from (4.42) that  $\alpha_{-1} = \alpha_{n-1} = 0$ , and thereby proving (4.46).

We proceed to use the recursion relations (4.45) and (4.46) to prove our theorem.

From (4.45) we have, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ , that

$$\left| \gamma_{2j+1}^{(r+1)} \right| \leq 2 \sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell| \left( |\gamma_{j+\ell+1}^{(r)}| + |\gamma_{j-\ell}^{(r)}| \right) \leq 4 \|\gamma^{(r)}\|_\infty \sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell| \quad (4.53)$$

and thus

$$\sup_j \left| \gamma_{2j+1}^{(r+1)} \right| \leq \left[ 4 \sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell| \right] \|\gamma^{(r)}\|_\infty, \quad r \in \mathbb{Z}_+, \quad (4.54)$$

whereas, similarly, from (4.46), we see, for  $j \in \mathbb{Z}$  and  $r \in \mathbb{Z}_+$ , that

$$\left| \gamma_{2j}^{(r+1)} \right| \leq \frac{1}{2} \|\gamma^{(r)}\|_\infty \left[ 1 + 4 \sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell| \right], \quad (4.55)$$

so that

$$\sup_j \left| \gamma_{2j}^{(r+1)} \right| \leq \left[ \frac{1}{2} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell| \right] \|\gamma^{(r)}\|_\infty, \quad r \in \mathbb{Z}_+. \quad (4.56)$$

Hence, if we define

$$\rho = \rho(\mathbf{a}) = \max \left\{ 4 \sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell|, \frac{1}{2} + 2 \sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell| \right\}, \quad (4.57)$$

it follows from (4.54), (4.56), together with (4.43) and (4.44), that (4.21) is satisfied, with

$\rho = \rho(\mathbf{a}) \in [0, 1]$  given by (4.57). ■

## 4.3 Examples

We proceed to apply our existence and convergence result of Theorem 4.8 to specific examples.

### 4.3.1 The cases $n = 2$ and $n = 3$ of Section 4.1

(a) First, we apply Theorem 4.8 to the family of mask symbols of Section 4.1, for the case  $n = 2$ , so that  $A(t|\cdot) \in \mathcal{A}_{1,1}$ ,  $t \in \mathbb{R}$ . From (4.1), (2.17) and (2.18), we obtain, for  $t \in \mathbb{R}$  and  $z \in \mathbb{C} \setminus \{0\}$ , the formula

$$\begin{aligned} A(t|z) &= (1-t)D_1(z) + tD_2(z) \\ &= -\frac{t}{16}z^{-3} + \left(\frac{1}{2} + \frac{t}{16}\right)z^{-1} + 1 + \left(\frac{1}{2} + \frac{t}{16}\right)z - \frac{t}{16}z^3, \end{aligned} \quad (4.58)$$

and thus, using (1.3), (4.58) and (4.42), we have

$$\alpha_0 = -\frac{t}{16}, \quad \alpha_j = 0, \quad j \neq 0, \quad t \in \mathbb{R}. \quad (4.59)$$

Noting, in the notation of Theorem 4.8, that we have here  $n = 2$ , and therefore, from (4.59), that in this case

$$\sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell| = \frac{1}{2} \left( |\alpha_{-1} + \alpha_0| + |\alpha_0 + \alpha_1| \right) = \frac{1}{2} \sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell|, \quad t \in \mathbb{R},$$

we immediately deduce that the conditions (4.43) and (4.44) of Theorem 4.8 hold if and only if (4.44) holds, i.e., if and only if  $2|\alpha_0| < \frac{1}{4}$ , or equivalently, according to (4.59), if and only if  $|t| < 2$ . It then follows from Theorem 4.8 that the family of masks  $A(t|\cdot)$ , as given by (4.58), satisfies all the conditions of Theorems 4.4 and 4.5, Corollary 4.6 and Theorem 4.7, if and only if  $|t| < 2$ .

In particular, the mask symbol (4.13) satisfies the conditions of Theorem 4.8, since there  $t = \frac{3}{2} < 2$ , and therefore, as was suggested by the graphical numerical evidence, the function plotted in Figure 4.1 is indeed the corresponding refinable function  $\phi$  of Theorem 4.4.

Now observe that Proposition 4.2 with  $\mathfrak{n} = 2$ , together with Theorem 2.2, imply the existence of a refinable function  $\phi$  and the convergence of the associated subdivision scheme  $S_a$  for  $t \in [-2, 1]$ . Hence, in particular, for  $t > 1$ , in which case, according to Proposition 4.2,  $A(t|\cdot)$  has two zeros on the unit circle in  $\mathbb{C}$ , and about which Theorem 2.2 is therefore inconclusive, our Theorem 4.8 actually extends the existence and convergence interval to  $t \in [-2, 2]$ .

It should be pointed out here that, according to explicit calculations by other authors (see [5, Example 3.1],[32]) with respect to more refined contractivity properties of the subdivision operator  $S_a$  than is captured by the inequality (4.21), the existence and convergence interval can actually be extended to  $t \in (-8, 8)$ . This is a substantial improvement on our analogous result, which merely yields the interval  $t \in (-2, 2)$ .

(b) Next, we consider the case  $\mathfrak{n} = 3$  in (4.1), for which, to our knowledge, an existence and convergence  $t$ -interval has not, as in (a) above, been previously investigated. In this case we have  $A(t|\cdot) \in \mathcal{A}_{1,2}$ ,  $t \in \mathbb{R}$ , and we obtain, from (4.1), (2.18) and (2.19), for  $t \in \mathbb{R}$  and  $z \in \mathbb{C} \setminus \{0\}$ , the formula

$$\begin{aligned} A(t|z) &= (1-t)D_2(z) + tD_3(z) \\ &= \frac{3t}{2^8}z^{-5} - \left(\frac{1}{16} + \frac{9t}{2^8}\right)z^{-3} + \left(\frac{9}{16} + \frac{6t}{2^8}\right)z^{-1} + 1 \\ &\quad + \left(\frac{9}{16} + \frac{6t}{2^8}\right)z - \left(\frac{1}{16} + \frac{9t}{2^8}\right)z^3 + \frac{3t}{2^8}z^5. \end{aligned} \quad (4.60)$$

Using (1.3), (4.60) and (4.42), we then calculate the values

$$\alpha_0 = -\left(\frac{1}{16} + \frac{3t}{2^8}\right), \quad \alpha_1 = \frac{3t}{2^8}, \quad \alpha_j = 0, \quad j \notin \mathbb{Z}_1, \quad t \in \mathbb{R}. \quad (4.61)$$

In the notation of Theorem 4.8, we now have  $\mathfrak{n} = 3$ , so that, using (4.61), we find that

$$\sum_{\ell \in \mathbb{Z}_{\mathfrak{n}-2}} |\alpha_\ell| = |\alpha_{-1} + \alpha_0| + |\alpha_1 + \alpha_2| \leq |\alpha_{-1} + \alpha_0| + |\alpha_0 + \alpha_1| + |\alpha_1 + \alpha_2| = \sum_{\ell \in \mathbb{Z}_{\mathfrak{n}-1}} |\alpha_{\ell-1} + \alpha_\ell|,$$

from which we immediately deduce that the conditions (4.43) and (4.44) of Theorem 4.8

hold if and only if (4.44) holds, i.e. if and only if

$$|\alpha_0| + |\alpha_0 + \alpha_1| + |\alpha_1| < \frac{1}{4}, \quad (4.62)$$

which we find, after substituting (4.61) into (4.62), to hold if and only if

$$|16 + 3t| + |3t| < 48, \quad (4.63)$$

or equivalently,  $|t - \frac{8}{3}| < 8$ , i.e.  $t \in (-\frac{32}{3}, \frac{16}{3})$ .

It then follows from Theorem 4.8 that the family of masks  $A(t|\cdot)$ , as given by (4.60), satisfies all the conditions of Theorems 4.4 and 4.5, Corollary 4.6 and Theorem 4.7, if and only if  $|t - \frac{8}{3}| < 8$ , i.e., if and only if  $t \in (-\frac{32}{3}, \frac{16}{3})$ .

For example, setting  $t = 5$  in (4.60), we obtain, for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$A(5|z) = \frac{1}{256} (15z^{-5} - 61z^{-3} + 174z^{-1} + 256 + 174z - 61z^3 + 15z^5), \quad (4.64)$$

for which, from the analysis above, there exists a refinable function, plotted in Figure 4.2 and a convergent subdivision scheme, as illustrated in Figure 4.3.

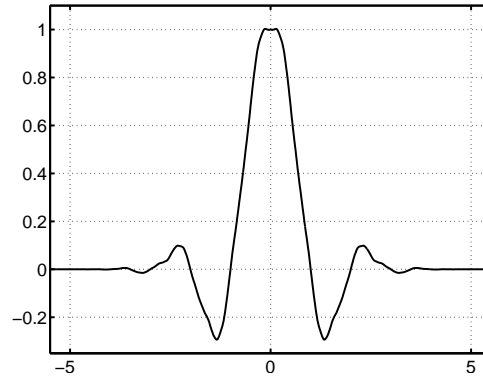


Figure 4.2: Refinable function with mask symbol (4.64)

Observe that, for the mask symbol  $A(t|\cdot)$  given by (4.60), Proposition 4.2 with  $n = 3$  and Theorem 2.2 imply the existence of a refinable function  $\phi$  and the convergence of the associated subdivision scheme  $S_a$  for  $t \in [-4, 1]$ . Hence, in particular for  $t > 1$ , in which case, according to Proposition 4.2,  $A(t|\cdot)$  has two zeros on the unit circle in  $\mathbb{C}$ , and about which Theorem 2.2 is therefore inconclusive, our Theorem 4.8 extends the existence and convergence interval to  $t \in [-4, \frac{16}{3})$ .

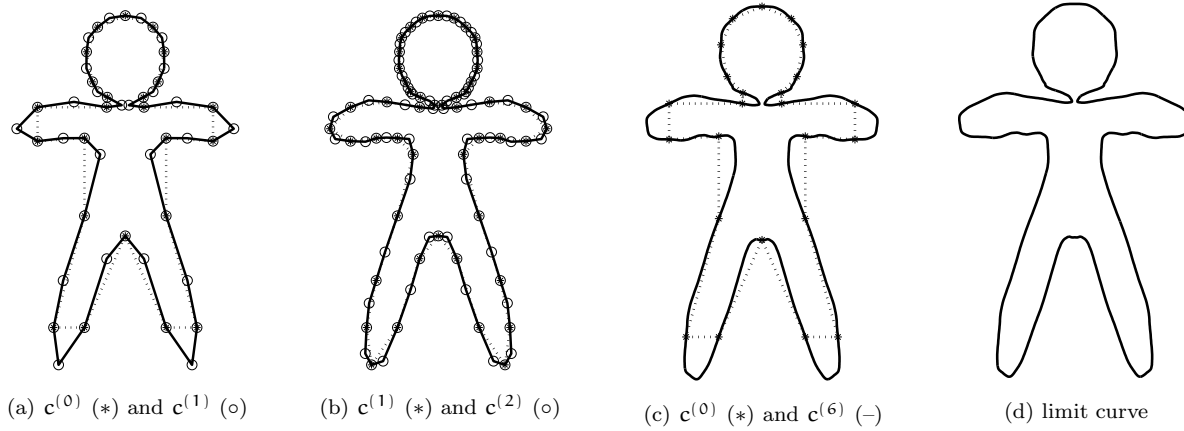


Figure 4.3: Illustration of subdivision with mask symbol (4.64)

### 4.3.2 The Dubuc–Deslauriers case

In this thesis, existence and convergence for the Dubuc–Deslauriers mask symbol  $\mathbf{A} = \mathbf{D}_n$  have already been established in Theorem 2.3 (see also the remark after Theorem 3.5). It is interesting to investigate whether the Dubuc–Deslauriers mask  $\mathbf{a} = \mathbf{d}_n$  actually satisfies the conditions (4.43) and (4.44) of Theorem 4.8, in which case we would then have an alternative existence and convergence proof. Since the values  $\mathbf{t} = 0$  and  $\mathbf{t} = 1$  are included in our convergence intervals  $(-2, 2)$  and  $(-\frac{32}{8}, \frac{16}{3})$  in Section 4.3.1, we can already conclude that the masks  $\mathbf{a} = \mathbf{d}_n$ ,  $n \in \{2, 3\}$  do indeed satisfy the conditions (4.43) and (4.44) of Theorem 4.8.

In general, with the notation

$$S_n = 4 \sum_{\ell \in \mathbb{Z}_{n-2}} |\alpha_\ell|, \quad T_n = 4 \sum_{\ell \in \mathbb{Z}_{n-1}} |\alpha_{\ell-1} + \alpha_\ell|, \quad (4.65)$$

with  $\{\alpha_\ell : \ell \in \mathbb{Z}_{n-1}\}$  defined by (4.42) and (2.12), and where here  $\mathbf{a} = \mathbf{d}_n$ , we wish to prove that

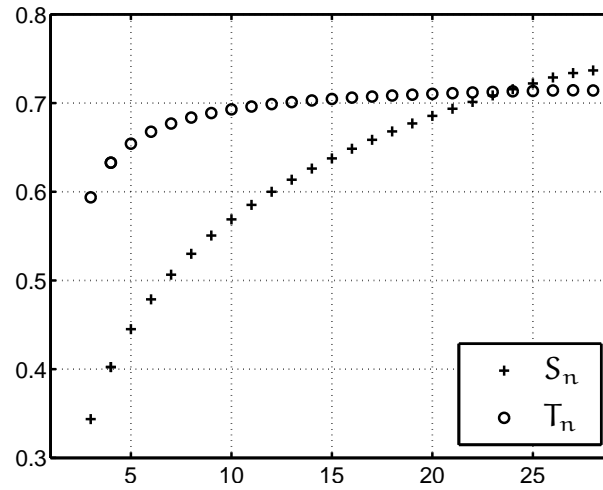
$$S_n < 1 \quad \text{and} \quad T_n < 1, \quad \text{for every integer } n \geq 2. \quad (4.66)$$

For the cases  $n = 2, 3, 4, 5$ , we use (2.16) in (4.42) to calculate the values in Table 4.1. It follows that the mask  $\mathbf{a} = \mathbf{d}_n$  satisfies the conditions (4.43) and (4.44) for  $n \in \{2, 3, 4, 5\}$ .

$n$	2	3	4	5
$S_n$	$\frac{1}{4}$	$\frac{11}{32}$	$\frac{103}{256}$	$\frac{555}{1247}$
$T_n$	$\frac{1}{2}$	$\frac{19}{32}$	$\frac{81}{128}$	$\frac{995}{1521}$

Table 4.1: Calculated values for the Dubuc–Deslauriers masks

Next, using a numerical technique based on (2.12) and (4.42), we obtain the graph in Figure 4.4 for  $n \in \{2, \dots, 28\}$ .

Figure 4.4: Numerical experiments with  $S_n$  and  $T_n$ .

On the basis of this numerical evidence, it seems plausible to conjecture that the Dubuc–Deslauriers mask  $\mathbf{a} = \mathbf{d}_n$  does indeed satisfy (4.66), and therefore the conditions (4.43) and (4.44) for all  $n \geq 2$ . A further investigation of this conjecture is in progress, and initial results are promising.

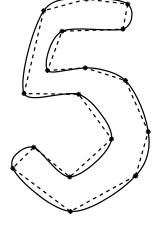




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# An extension of Dubuc–Deslauriers subdivision

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We present here a further method of generating a symbol  $A \in \mathcal{A}_{\mu, \nu}$ , i.e. a symmetric, interpolatory mask symbol of degree at most  $2(\mu + \nu) - 1$ , and with, according to Proposition 2.5, a zero of order at least  $2\nu$  at  $z = -1$ . This method, as first introduced in [22], is based on a local interpolation scheme.

The basic idea described here is to add in a symmetric function sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ , to a basis for the polynomial space  $\pi_{2\nu-1}$ , and then design a subdivision scheme by constructing a sequence of fundamental interpolants, which, when evaluated at  $\frac{1}{2}$ , as in the Dubuc–Deslauriers construction (2.12), then yields a symmetric, interpolatory subdivision scheme that can be considered as an extension of Dubuc–Deslauriers subdivision. Since we are adding in functions to a basis for the polynomial space  $\pi_{2\nu-1}$ , our subdivision scheme will then also automatically satisfy (2.38).

In Section 5.1 below, we provide a sufficient condition on the above-mentioned sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  to ensure that the subdivision mask of the newly generated subdivision scheme is in  $\mathcal{A}_{\mu, \nu}$ . Then, in Section 5.2, we choose the sequence of functions  $\{f_j : j \in \mathbb{J}_\mu\}$  as truncated powers and study the resulting simplifications of the general theory. Also, in specific examples, we investigate the issue of the existence of a corresponding refinable function, and the convergence of the associated subdivision scheme.

## 5.1 The general construction method

In this section we present sufficient conditions on the above-mentioned function sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  in order to generate a subdivision mask symbol  $\mathbf{A}$  in  $\mathcal{A}_{\mu,\nu}$ .

For  $n \in \mathbb{Z}_+$ , let  $x_0 < x_1 < \dots < x_n$  be a sequence of points in  $\mathbb{R}$ , and suppose  $f$  is a given real-valued function with domain  $D_f \subset \mathbb{R}$  such that  $[x_0, x_n] \subset D_f$ . We shall denote by  $[x_0, \dots, x_n]f$  the  $n$ -th order divided difference of  $f$  with respect to the points  $x_0, \dots, x_n$  (see e.g. [56, Chapter I, Section 2.2]). With the conventional definition  $[x]f := f(x)$ ,  $x \in [x_0, x_n]$ , we then have, for  $0 \leq j < j + \ell + 1 \leq n$ , the recursion formula

$$[x_j, \dots, x_{j+\ell+1}]f = \frac{[x_{j+1}, \dots, x_{j+\ell+1}]f - [x_j, \dots, x_{j+\ell}]f}{x_{j+\ell+1} - x_j}, \quad (5.1)$$

that can be used recursively to compute the divided difference  $[x_0, \dots, x_n]f$ . Also, if  $f$  has  $n$  continuous derivatives on the interval  $[x_0, x_n]$ , then there exists a point  $\xi \in (x_0, x_n)$  such that

$$[x_0, \dots, x_n]f = \frac{1}{n!}f^{(n)}(\xi). \quad (5.2)$$

We shall rely on the following lemma to develop the theory.

**LEMMA 5.1** *For  $m \in \mathbb{N}$ , let  $x_0 < x_1 < \dots < x_m$  be a sequence of points in  $\mathbb{R}$  and let  $g_j : [x_0, x_m] \rightarrow \mathbb{R}$ ,  $j \in \mathbb{Z}_m$ , denote a sequence of functions. Then the determinant sequence*

$$D_k = \det(A_k), \quad k \in \mathbb{Z}_m, \quad (5.3)$$

*where the matrices  $\{A_k : k \in \mathbb{Z}_m\}$  are defined by*

$$(A_k)_{ij} = [x_i, \dots, x_{\min\{i+k, m\}}]g_j, \quad i, j \in \mathbb{Z}_m, \quad (5.4)$$

*satisfies the relation*

$$D_k = \left[ \prod_{j \in \mathbb{Z}_{m-1-k}} (x_j - x_{j+k+1}) \right] D_{k+1}, \quad k \in \mathbb{Z}_{m-1}. \quad (5.5)$$

**Proof.** The result (5.5) is obtained by performing successive row subtractions on the first  $(\mathfrak{m} - k)$  rows of the determinant  $D_k$  and using (5.1). ■

The following existence result can now be proved.

**THEOREM 5.2** *For  $\mu, \nu \in \mathbb{N}$ , suppose  $f_j : [-\mu - \nu + 1, \mu + \nu] \rightarrow \mathbb{R}$ ,  $j \in \mathbb{J}_\mu$ , is a sequence of functions such that the  $2\mu \times 2\mu$  matrix  $X$  defined by*

$$(X)_{ij} = [-\nu + i, \dots, \nu + i] f_j, \quad i, j \in \mathbb{J}_\mu, \quad (5.6)$$

*is invertible. Then the linear space  $S_{\mu, \nu}$  defined by*

$$S_{\mu, \nu} := \text{span}\{f_j : j \in \mathbb{J}_\mu\} \oplus \pi_{2\nu-1} \quad (5.7)$$

*has dimension  $(2\mu + 2\nu)$ , and there exists a unique fundamental sequence  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\} \subset S_{\mu, \nu}$  such that*

$$R_j(k) = \delta_{j,k}, \quad j, k \in \mathbb{J}_{\mu+\nu}. \quad (5.8)$$

*Also,*

$$\sum_{j \in \mathbb{J}_{\mu+\nu}} s(j) R_j = s, \quad s \in S_{\mu, \nu}. \quad (5.9)$$

*If, moreover,*

$$f_j = f_{1-j}(1 - \cdot), \quad j \in \mathbb{J}_\mu, \quad (5.10)$$

*then*

$$R_j = R_{1-j}(1 - \cdot), \quad j \in \mathbb{J}_{\mu+\nu}. \quad (5.11)$$

**Proof.** The existence of a unique fundamental sequence  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\} \subset S_{\mu, \nu}$  satisfying the interpolatory condition (5.8) will follow if we can prove, for each  $j \in \mathbb{J}_{\mu+\nu}$ , the existence of a unique sequence  $\{\beta_{ji} : i = -\mu + 1, \dots, \mu + 2\nu\}$ , such that

$$\sum_{i \in \mathbb{J}_\mu} \beta_{ji} f_i(k) + \sum_{i \in \{\mu + \mathbb{N}_{2\nu}\}} \beta_{ji} k^{i-\mu-1} = \delta_{j,k}, \quad k \in \mathbb{J}_{\mu+\nu}, \quad (5.12)$$

or, equivalently, if we can prove that the determinant  $D$  of the coefficient matrix of the linear system (5.12), which is given by

$$D = \begin{vmatrix} f_{-\mu+1}(-\mu-\nu+1) & \cdots & f_{\mu}(-\mu-\nu+1) & 1 & (-\mu-\nu+1) & \cdots & (-\mu-\nu+1)^{2\nu-1} \\ f_{-\mu+1}(-\mu-\nu+2) & \cdots & f_{\mu}(-\mu-\nu+2) & 1 & (-\mu-\nu+2) & \cdots & (-\mu-\nu+2)^{2\nu-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_{-\mu+1}(\mu+\nu) & \cdots & f_{\mu}(\mu+\nu) & 1 & (\mu+\nu) & \cdots & (\mu+\nu)^{2\nu-1} \end{vmatrix} \quad (5.13)$$

is nonzero.

Now consider the determinant sequence  $\{D_k : k \in \mathbb{Z}_m\}$  of Lemma 5.1, with the choices  $m = 2\mu + 2\nu - 1$ ,  $x_i = -\mu - \nu + 1 + i$ ,  $i \in \mathbb{Z}_m$ , and

$$g_j = \begin{cases} f_{j-\mu+1}, & j \in \mathbb{Z}_{2\mu-1}, \\ (\cdot)^{j-2\mu}, & j \in \{2\mu + \mathbb{Z}_{2\nu-1}\}, \end{cases} \quad (5.14)$$

where, for  $n \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}$ , we use the notation  $\{k + \mathbb{Z}_n\}$  to denote the set  $\{k, k+1, \dots, k+n\}$ . It then follows from (5.13), (5.14), (5.3) and (5.4), that  $D = D_0$ . Hence, from (5.5),  $D$  is nonzero if and only if

$$D_{2\nu} \neq 0. \quad (5.15)$$

Using also the property (5.2) of divided differences, we deduce that the determinant  $D_{2\nu}$  has the structure

$$D_{2\nu} = \begin{vmatrix} [-\mu-\nu+1, \dots, -\mu+\nu+1] f_{-\mu+1} & \cdots & [-\mu-\nu+1, \dots, -\mu+\nu+1] f_{\mu} & 0 & 0 & \cdots & 0 & 0 \\ [-\mu-\nu+2, \dots, -\mu+\nu+2] f_{-\mu+1} & \cdots & [-\mu-\nu+2, \dots, -\mu+\nu+2] f_{\mu} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ [\mu-\nu, \dots, \mu+\nu] f_{-\mu+1} & \cdots & [\mu-\nu, \dots, \mu+\nu] f_{\mu} & 0 & 0 & \cdots & 0 & 0 \\ \times & \cdots & \times & 0 & 0 & \cdots & 0 & 1 \\ \times & \cdots & \times & 0 & 0 & \cdots & 1 & \times \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \times & \cdots & \times & 0 & 1 & \cdots & \times & \times \\ \times & \cdots & \times & 1 & \times & \cdots & \times & \times \end{vmatrix}. \quad (5.16)$$

By exploiting the structure of the zeros and ones in the last  $2\nu$  columns of  $D_{2\nu}$  in (5.16), we now expand  $D_{2\nu}$  with respect to these columns to deduce from (5.6) that  $D_{2\nu} = \det(X) \neq 0$ , and from which it then follows that (5.15) does indeed hold.

For each fixed  $j \in \mathbb{J}_{\mu+\nu}$ , the unique solution  $\{\beta_{ji} : i = -\mu + 1, \dots, \mu + 2\nu\}$  of the  $(2\mu + 2\nu) \times (2\mu + 2\nu)$  linear system (5.12) then yields the fundamental function sequence  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\} \subset S_{\mu,\nu}$ , as given by

$$R_j = \sum_{i \in \mathbb{J}_\mu} \beta_{ji} f_i + \sum_{i \in \{\mu + \mathbb{N}_{2\nu}\}} \beta_{ji} (\cdot)^{i-\mu-1}, \quad j \in \mathbb{J}_{\mu+\nu}, \quad (5.17)$$

and which satisfies the interpolation condition (5.8).

According to (5.7),  $\dim(S_{\mu,\nu}) \leq 2\mu + 2\nu$ . But the interpolatory condition (5.8) implies that the set  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\} \subset S_{\mu,\nu}$  is linearly independent, so that  $\dim(S_{\mu,\nu}) \geq 2\mu + 2\nu$ . Hence  $\dim(S_{\mu,\nu}) = 2\mu + 2\nu$ , and it immediately follows that the set  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\}$  is a basis for  $S_{\mu,\nu}$ .

To prove (5.9), suppose  $s \in S_{\mu,\nu}$ . Since  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\}$  is a basis for  $S_{\mu,\nu}$ , there exists a unique sequence  $\{\beta_j : j \in \mathbb{J}_{\mu+\nu}\} \subset \mathbb{R}$  such that  $s = \sum_{j \in \mathbb{J}_{\mu+\nu}} \beta_j R_j$ . But then (5.8) yields  $\beta_j = s(j)$ ,  $j \in \mathbb{J}_{\mu+\nu}$ , thereby proving (5.9).

Finally, suppose the symmetry condition (5.10) is satisfied. With  $\{\ell_{\nu,j} : j \in \mathbb{J}_\nu\}$  denoting the Lagrange fundamental polynomials in  $\pi_{2\nu-1}$  as defined by (2.8), we now define the sequence  $\{w_j : j \in \mathbb{J}_{\mu+\nu}\} \subset S_{\mu,\nu}$  by

$$w_j = \begin{cases} f_{j+\nu} & j \in \{-\nu + \mathbb{J}_\mu\}, \\ \ell_{\nu,j-\mu} & j \in \{\mu + \mathbb{J}_\nu\}, \end{cases} \quad (5.18)$$

where, for  $n \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}$ , we use the notation  $\{k + \mathbb{J}_n\}$  for the set  $\{k - n + 1, k - n + 2, \dots, k + n\}$ . Since  $\{\ell_{\nu,j} : j \in \mathbb{J}_\nu\}$  is a basis for  $\pi_{2\nu-1}$ , it follows from (5.7) and (5.18), together with the fact that  $\dim(S_{\mu,\nu}) = 2\mu + 2\nu$ , that  $\{w_j : j \in \mathbb{J}_{\mu+\nu}\}$  is also a basis for  $S_{\mu,\nu}$ . We claim that

$$w_j = w_{1-j}(1 - \cdot), \quad j \in \mathbb{J}_{\mu+\nu}. \quad (5.19)$$

To prove this statement, we first note from (5.18) and the symmetry condition (5.10), that (5.19) holds for  $j \in \{-\nu + \mathbb{J}_\mu\}$ . Next, we observe that the Lagrange fundamental polynomials  $\{\ell_{\nu,j} : j \in \mathbb{J}_\nu\}$  are symmetric about  $\frac{1}{2}$  since, for  $j \in \mathbb{J}_\nu$ , we have, from (2.8),

that

$$\begin{aligned}
 \ell_{\nu, 1-j}(1-\cdot) &= \prod_{1-j \neq k \in \mathbb{J}_{\nu}} \frac{1-\cdot-k}{1-j-k} \\
 &= \prod_{j \neq k \in \mathbb{J}_{\nu}} \frac{1-\cdot-(1-k)}{1-j-(1-k)} \\
 &= \prod_{j \neq k \in \mathbb{J}_{\nu}} \frac{\cdot-k}{j-k} = \ell_{\nu, j}.
 \end{aligned}$$

Thus (5.19) also holds for  $j \in \{\mu + \mathbb{J}_{2\nu}\}$ . With the definition

$$\tilde{R}_j = R_{1-j}(1-\cdot), \quad j \in \mathbb{J}_{\mu+\nu}, \quad (5.20)$$

we have  $\tilde{R}_j : [\mu - \nu + 1, \mu + \nu] \rightarrow \mathbb{R}$ ,  $j \in \mathbb{J}_{\mu+\nu}$ , by virtue of the fact that  $R_j \in S_{\mu, \nu}$ , i.e.  $R_j : [\mu - \nu + 1, \mu + \nu] \rightarrow \mathbb{R}$ ,  $j \in \mathbb{J}_{\mu+\nu}$ . Moreover, (5.8) and (5.20) imply that

$$\tilde{R}_j(k) = \delta_{j,k}, \quad j, k \in \mathbb{J}_{\mu+\nu}. \quad (5.21)$$

Since  $\{w_j : j \in \mathbb{J}_{\mu+\nu}\}$  is a basis for  $S_{\mu, \nu}$ , there exists, for every  $j \in \mathbb{J}_{\mu+\nu}$ , a sequence  $\{c_{j,k} : k \in \mathbb{J}_{\mu+\nu}\}$  such that  $R_j = \sum_{k \in \mathbb{J}_{\mu+\nu}} c_{j,k} w_k$ , and thus, using also (5.19), we get

$$\tilde{R}_j = \sum_{k \in \mathbb{J}_{\mu+\nu}} c_{1-j,k} w_k(1-\cdot) = \sum_{k \in \mathbb{J}_{\mu+\nu}} c_{1-j, 1-k} w_k, \quad j \in \mathbb{J}_{\mu+\nu},$$

from which we deduce that  $\tilde{R}_j \in S_{\mu, \nu}$ ,  $j \in \mathbb{J}_{\mu+\nu}$ . Since also (5.21) holds, and since  $\{R_j : j \in \mathbb{J}_{\mu+\nu}\}$  is the unique function sequence in  $S_{\mu, \nu}$  satisfying (5.8), it follows that  $\tilde{R}_j = R_j$ ,  $j \in \mathbb{J}_{\mu+\nu}$ , which, together with (5.20), yields the desired symmetry result (5.11).  $\blacksquare$

Suppose, for given positive integers  $\mu$  and  $\nu$ , we are given a sequence  $\{f_j : j \in \mathbb{J}_{\mu}\}$  that satisfies the conditions of Theorem 5.2. Analogous to the Dubuc–Deslauriers definition (2.12), we now define the mask

$$\left. \begin{aligned}
 a_{2j} &= \delta_j, & j \in \mathbb{Z}, \\
 a_{1-2j} &= R_j\left(\frac{1}{2}\right), & j \in \mathbb{J}_{\mu+\nu}, \\
 a_j &= 0, & j \notin \mathbb{J}_{\mu+\nu}.
 \end{aligned} \right\} \quad (5.22)$$

The following result then holds.

**THEOREM 5.3** *If the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  is a sequence as in Theorem 5.2, and such that the symmetry condition (5.10) is also satisfied, then the Laurent polynomial  $A$  defined by (1.3) and (5.22) belongs to the class  $\mathcal{A}_{\mu,\nu}$ .*

**Proof.** Referring to Definition 2.4, we first observe from (5.22) that the Laurent polynomial  $A$  has degree at most  $2(\mu + \nu) - 1$ . Next, from the top line of (5.22), together with Proposition 2.1, we conclude that (2.37) holds.

To prove that the symmetry condition (2.36) is satisfied, we observe from the middle line of (5.22), together with (5.11), that, for  $j \in \mathbb{J}_{\mu+\nu}$ ,

$$a_{1-2j} = R_j \left( \frac{1}{2} \right) = R_{1-j} \left( \frac{1}{2} \right) = a_{1-2(1-j)} = a_{2j-1}.$$

Hence, noting also the top and bottom equations in (5.22), we get  $a_{-j} = a_j$ ,  $j \in \mathbb{Z}$ , which is equivalent to the mask symbol symmetry condition (2.36).

Finally, to prove that the polynomial filling property (2.38) holds, let  $p \in \pi_{2\nu-1}$ . Then, from the top line of (5.22), we get

$$\sum_k a_{2j-2k} p(k) = p(j), \quad j \in \mathbb{Z}. \quad (5.23)$$

Next, for  $j \in \mathbb{Z}$ , we define  $q_j = p(j + \cdot)$ , so that also,  $q_j \in \pi_{2\nu-1}$ . But the definition (5.7) shows that  $\pi_{2\nu-1} \subset S_{\mu,\nu}$ , so that  $q \in S_{\mu,\nu}$ . Hence, using (5.22) and (5.9), we get, for  $j \in \mathbb{Z}$ , that

$$\begin{aligned} \sum_k a_{2j+1-2k} p(k) &= \sum_k a_{2k+1} p(j-k) \\ &= \sum_k a_{1-2k} p(j+k) \\ &= \sum_{k \in \mathbb{J}_{\mu+\nu}} a_{1-2k} p(j+k) \\ &= \sum_{k \in \mathbb{J}_{\mu+\nu}} q_j(k) R_k \left( \frac{1}{2} \right) = q_j \left( \frac{1}{2} \right) = p \left( \frac{2j+1}{2} \right), \end{aligned}$$

which, together with (5.23), implies that (2.38) is satisfied. Hence, according to Definition 2.4, we have  $A \in \mathcal{A}_{\mu,\nu}$ . ■

Observe, from (2.12) and (5.22), that the Dubuc–Deslauriers subdivision scheme  $\mathbf{S}_{\mathbf{d}_n}$  can be interpreted as the additional case  $\mu = 0$  and  $\nu = n$  in the context of Theorem 5.3, and that the subdivision scheme (5.22) can therefore be regarded as an extension of Dubuc–Deslauriers subdivision, as obtained by adding in, according to Theorems 5.2 and 5.3, a sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  to the polynomial space  $\pi_{2\nu-1}$  to form the enlarged space  $\mathbf{S}_{\mu,\nu}$ .

## 5.2 The resulting Dubuc–Deslauriers expansion

From Theorem 2.7, we know that any mask symbol  $A \in \mathcal{A}_{\mu,\nu}$  can be written as a (unique) linear combination of Dubuc–Deslauriers symbols  $\{D_{\nu+j} : j \in \mathbb{Z}_\mu\}$ . We proceed to determine, for the mask symbol  $A$  defined by (1.3) and (5.22), the coefficients of the linear combination (2.65) in terms of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ . Since the Dubuc–Deslauriers mask symbols  $\{D_{\nu+j} : j \in \mathbb{Z}_\mu\}$  in this expansion can be explicitly computed from (2.16) and (2.13), and since the coefficient sequence  $\{t_j : j \in \mathbb{Z}_\mu\}$  will be shown below to be obtainable as the unique solution of a  $(\mu+1) \times (\mu+1)$  linear system, we would then have established an explicit computational method for  $A$  that is more efficient than the one based on the expansion (5.17) and which depends on solving the  $(2\mu+2\nu) \times (2\mu+2\nu)$  linear system (5.12).

Suppose therefore that  $\{t_j : j \in \mathbb{Z}_\mu\}$  is, according to Theorems 2.7 and 5.3, the unique sequence in  $\mathbb{R}$  such that  $\sum_{j \in \mathbb{Z}_\mu} t_j = 1$  and  $A = \sum_{j \in \mathbb{Z}_\mu} t_j D_{\nu+j}$ . Thus, from (1.3) and (5.22), together with (2.12) and (2.13), we have for  $z \in \mathbb{C} \setminus \{0\}$  that

$$\begin{aligned} 1 + \sum_{k \in \mathbb{J}_{\mu+\nu}} R_k \left( \frac{1}{2} \right) z^{1-2k} = A(z) &= 1 + \sum_{k \in \mathbb{J}_{\mu+\nu}} a_{1-2k} z^{1-2k} \\ &= 1 + \sum_{k \in \mathbb{J}_{\mu+\nu}} \left[ \sum_{j \in \mathbb{Z}_\mu} t_j d_{\nu+j, 1-2k} \right] z^{1-2k}, \end{aligned}$$

and thus

$$\sum_{k \in \mathbb{J}_{\mu+\nu}} R_k \left( \frac{1}{2} \right) z^{1-2k} = \sum_{k \in \mathbb{J}_{\mu+\nu}} \left[ \sum_{j \in \mathbb{Z}_\mu} t_j d_{\nu+j, 1-2k} \right] z^{1-2k}, \quad z \in \mathbb{C} \setminus \{0\}, \quad (5.24)$$



so that

$$R_k \left( \frac{1}{2} \right) = \sum_{j \in \mathbb{Z}_\mu} t_j d_{v+j, 1-2k}, \quad k \in \mathbb{J}_{\mu+v}. \quad (5.25)$$

For  $i \in \mathbb{J}_\mu$ , we have from (5.7) that  $f_i \in S_{\mu, v}$ , so that (5.9) and (5.25) yield

$$\begin{aligned} f_i \left( \frac{1}{2} \right) &= \sum_{k \in \mathbb{J}_{\mu+v}} f_i(k) R_k \left( \frac{1}{2} \right) = \sum_{k \in \mathbb{J}_{\mu+v}} f_i(k) \left[ \sum_{j \in \mathbb{Z}_\mu} t_j d_{v+j, 1-2k} \right] \\ &= \sum_{j \in \mathbb{Z}_\mu} \left[ \sum_{k \in \mathbb{J}_{\mu+v}} f_i(k) d_{v+j, 1-2k} \right] t_j \\ &= \sum_{j \in \mathbb{Z}_\mu} \left[ \sum_{k \in \mathbb{J}_{v+j}} f_i(k) \ell_{v+j, k} \left( \frac{1}{2} \right) \right] t_j, \end{aligned} \quad (5.26)$$

by virtue of (2.12). Using also the fact that  $\sum_{j \in \mathbb{J}_\mu} t_j = 1$ , we deduce from (5.26) that

$$\sum_{j \in \mathbb{Z}_\mu} \left[ f_i \left( \frac{1}{2} \right) - \sum_{k \in \mathbb{J}_{v+j}} f_i(k) \ell_{v+j, k} \left( \frac{1}{2} \right) \right] t_j = 0, \quad i \in \mathbb{J}_\mu. \quad (5.27)$$

A standard error expression in polynomial interpolation (see e.g. [56, Chapter I, Theorem 2.8]) gives

$$f_i \left( \frac{1}{2} \right) - \sum_{k \in \mathbb{J}_{v+j}} f_i(k) \ell_{v+j, k} \left( \frac{1}{2} \right) = \rho_{v+j} \left[ \frac{1}{2}, -v-j+1, \dots, v+j \right] f_i, \quad j \in \mathbb{Z}_\mu, \quad i \in \mathbb{J}_\mu, \quad (5.28)$$

where

$$\rho_k = \prod_{m \in \mathbb{J}_k} \left( \frac{1}{2} - m \right) = \frac{(-1)^k}{2^{4k-2}} \left[ \frac{(2k-1)!}{(k-1)!} \right]^2, \quad k \in \mathbb{N}, \quad (5.29)$$

as follows immediately from (2.15), so that also the recursion formula

$$\rho_{k+1} = - \left( \frac{2k+1}{2} \right)^2 \rho_k, \quad k \in \mathbb{N}, \quad (5.30)$$

is satisfied. Combining (5.27), (5.28), the symmetry (5.10) of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ ,

and the fact that  $\sum_{j \in \mathbb{Z}_\mu} t_j = 1$ , we deduce that, in the vector notation

$$\mathbf{t} = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_\mu \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (5.31)$$

and with the  $(\mu + 1) \times (\mu + 1)$  matrix  $B$  given by

$$(B)_{ij} = \begin{cases} 1, & i = 0 \\ \rho_{\nu+j} \left[ \frac{1}{2}, -\nu-j+1, \dots, \nu+j \right] f_i, & i \in \mathbb{N}_\mu, \end{cases} \quad j \in \mathbb{Z}_\mu, \quad (5.32)$$

we have

$$Bt = \mathbf{b}. \quad (5.33)$$

We have therefore established, as was done in [22], the following result.

**THEOREM 5.4** *For the mask symbol  $A \in \mathcal{A}_{\mu,\nu}$  of Theorem 5.3, the corresponding unique coefficient sequence  $\mathbf{t} = \{t_j : j \in \mathbb{Z}_\mu\}$  in (2.65) of Theorem 2.7, satisfies the  $(\mu+1) \times (\mu+1)$  linear system (5.33).*

We proceed to prove here that the matrix  $B$ , as given by (5.32) and (5.29), is invertible, from which, together with Theorem 5.4, it will then follow that the unique solution  $\mathbf{t}$  of the linear system (5.33) yields, through its definition in (5.31), the desired coefficient sequence  $\{t_j : j \in \mathbb{Z}_\mu\}$  in the Dubuc–Deslauriers expansion (2.65) of the mask symbol  $A$  of Theorem 5.3.

To this end, we first prove the following implication of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  satisfying the conditions of Theorem 5.2.

**PROPOSITION 5.5** *If, in Theorem 5.2, the symmetry condition (5.10) is also satisfied by the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ , then the  $\mu \times \mu$  matrix  $H$  defined by*

$$(H)_{jk} = h_{jk} = [-\nu-k+1, \dots, \nu-k+1]f_j + [-\nu+k, \dots, \nu+k]f_j, \quad j, k \in \mathbb{N}_\mu, \quad (5.34)$$

*is invertible.*

**Proof.** According to Theorem 5.2, we know that the  $2\mu \times 2\mu$  matrix  $X$ , where

$$(X)_{jk} = x_{jk} = [-\mu-\nu+j, \dots, \nu-\mu+j]f_{-\mu+k}, \quad j, k \in \mathbb{N}_{2\mu}, \quad (5.35)$$

satisfies  $\det(X) \neq 0$ . Now define the two  $2\mu \times 2\mu$  matrices  $U$  and  $V$  by

$$(U)_{jk} = u_{jk} = \begin{cases} x_{jk} - x_{j,2\mu+1-k}, & k \in \mathbb{N}_\mu, \\ x_{jk}, & k \in \{\mu + \mathbb{N}_\mu\}, \end{cases} \quad j \in \mathbb{N}_{2\mu}, \quad (5.36)$$

and

$$(V)_{jk} = v_{jk} = \begin{cases} u_{jk} + u_{2\mu+1-j,k}, & j \in \mathbb{N}_\mu, \\ u_{jk}, & j \in \{\mu + \mathbb{N}_\mu\}, \end{cases} \quad k \in \mathbb{N}_{2\mu}, \quad (5.37)$$

so that  $\det(X) = \det(U) = \det(V)$ , and therefore  $\det(V) \neq 0$ .

Next, we use (5.37), (5.36) and (5.35), together with (5.10) and the fact that a divided difference is a symmetric function of its arguments, to deduce that, for  $j, k \in \mathbb{N}_\mu$ , we have

$$\begin{aligned} v_{jk} &= u_{jk} + u_{2\mu+1-j,k} \\ &= g_{jk} - g_{j,2\mu+1-k} + g_{2\mu+1-j,k} - g_{2\mu+1-j,2\mu+1-k} \\ &= [-\mu-\nu+j, \dots, \nu-\mu+j]f_{-\mu+k} - [-\mu-\nu+j, \dots, \nu-\mu+j]f_{\mu+1-k} \\ &\quad + [-\nu+\mu+1-j, \dots, \mu+\nu+1-j]f_{-\mu+k} - [-\nu+\mu+1-j, \dots, \mu+\nu+1-j]f_{\mu+1-k} \\ &= [-\nu+\mu+1-j, \dots, \mu+\nu+1-j]f_{\mu+1-k} - [-\mu-\nu+j, \dots, \nu-\mu+j]f_{\mu+1-k} \\ &\quad + [-\mu-\nu+j, \dots, \nu-\mu+j]f_{\mu+1-k} - [-\nu+\mu+1-j, \dots, \mu+\nu+1-j]f_{\mu+1-k} \\ &= 0, \end{aligned}$$

and thus, from a standard block-partitioned matrix result for determinants (see e.g. [34, Chapter 2, Section 5]), we obtain

$$\det(V) = (-1)^\mu \det(M) \det(N), \quad (5.38)$$

where the  $\mu \times \mu$  matrices  $M$  and  $N$  are defined by

$$(M)_{jk} = m_{jk} = v_{j,\mu+k}, \quad j, k \in \mathbb{N}_\mu, \quad (5.39)$$

$$(N)_{jk} = n_{jk} = v_{\mu+j,k}, \quad j, k \in \mathbb{N}_\mu. \quad (5.40)$$

Since  $\det(V) \neq 0$ , we conclude from (5.38) that  $\det(M) \neq 0$ . But, from (5.39), (5.37), (5.36) and (5.35), we find that, for  $j, k \in \mathbb{N}_\mu$ ,

$$\begin{aligned} m_{jk} &= u_{j,\mu+k} + u_{2\mu+1-j,\mu+k} \\ &= x_{j,\mu+k} + x_{2\mu+1-j,\mu+k} \\ &= [-\mu-\nu+j, \dots, \nu-\mu+j]f_k + [-\nu+\mu+1-j, \dots, \mu+\nu+1-j]f_k. \end{aligned} \quad (5.41)$$

Next we define  $P = M^T$ , so that, from (5.41),

$$(P)_{jk} = [-\mu-\nu+k, \dots, \nu-\mu+k]f_j + [-\nu+\mu+1-k, \dots, \mu+\nu+1-k]f_j, \quad j, k \in \mathbb{N}_\mu. \quad (5.42)$$

Then  $\det(P) = \det(M)$ , and thus  $\det(P) \neq 0$ . Finally, observe from (5.34) and (5.42) that

$$(H)_{jk} = (P)_{j, \mu+1-k}, \quad j, k \in \mathbb{N}_\mu,$$

and thus  $\det(H) = (-1)^{[\mu/2]} \det(P)$ . Hence  $\det(H) \neq 0$ , i.e.  $H$  is invertible.  $\blacksquare$

Using Proposition 5.5, we can now prove the invertibility of the matrix  $B$ .

**THEOREM 5.6** *If, in Theorem 5.2, the symmetry condition (5.10) is also satisfied by the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ , then the matrix  $B$  defined by (5.32) and (5.29) is invertible.*

**Proof.** First, we use (5.28) in the definition (5.32) of the matrix  $B$ , to deduce, by subtracting successive columns in  $\det(B)$ , that  $\det(B) = (-1)^\mu \det(C)$ , where  $C$  is the  $\mu \times \mu$  matrix defined by

$$(C)_{jk} = p_{jk} \left(\frac{1}{2}\right) - p_{j, k-1} \left(\frac{1}{2}\right), \quad j, k \in \mathbb{N}_\mu, \quad (5.43)$$

and with  $p_{jk}$  denoting, for  $k \in \mathbb{N}_\mu$ , the (unique) polynomial in  $\pi_{2(\nu+k)-1}$  that interpolates  $f_j$  on the set  $\mathbb{J}_{\nu+k}$ . Hence it will suffice to prove that  $\det(C) \neq 0$ , for then  $\det(B) \neq 0$ , and thus  $B$  is invertible.

From the Newton divided difference formula in polynomial interpolation (see e.g. [56, Chapter I, Theorem 2.6]), we have, for  $j, k \in \mathbb{N}_\mu$ , that

$$p_{jk} - p_{j, k-1} = [-\nu-k+1, \dots, \nu+k-1]f_j \prod_{\ell \in \mathbb{J}_{\nu+k-1}} (\cdot - \ell) + [-\nu-k+1, \dots, \nu+k]f_j \prod_{\ell = -\nu-k+1}^{\nu+k-1} (\cdot - \ell),$$

which, together with (5.43), yields

$$(C)_{jk} = \rho_{\nu+k-1} \left[ (\nu + k - \frac{1}{2}) [-\nu-k+1, \dots, \nu+k]f_j + [-\nu-k+1, \dots, \nu+k-1]f_j \right], \quad j, k \in \mathbb{N}_\mu, \quad (5.44)$$

where  $\rho_k$  is defined by (5.29). But, using (5.1), we have

$$[-\nu-k+1, \dots, \nu+k]f_j = \frac{1}{2\nu + 2k - 1} \left( [-\nu-k+2, \dots, \nu+k]f_j - [-\nu-k+1, \dots, \nu+k-1]f_j \right), \quad j, k \in \mathbb{N}_\mu,$$

which, together with (5.44), yields

$$(C)_{jk} = \frac{1}{2} \rho_{\nu+k-1} \left( [-\nu-k+1, \dots, \nu+k-1] f_j + [-\nu-k+2, \dots, \nu+k] f_j \right), \quad j, k \in \mathbb{N}_\mu. \quad (5.45)$$

Hence,

$$\det(C) = \frac{1}{2^\mu} \left[ \prod_{k \in \mathbb{N}_\mu} \rho_{\nu+k-1} \right] \det(E), \quad (5.46)$$

where  $E$  is the  $\mu \times \mu$  matrix defined by

$$(E)_{jk} = e_{jk} = [-\nu-k+1, \dots, \nu+k-1] f_j + [-\nu-k+2, \dots, \nu+k] f_j, \quad j, k \in \mathbb{N}_\mu. \quad (5.47)$$

It will therefore suffice to prove that  $\det(E) \neq 0$ , since then (5.46) gives  $\det(C) \neq 0$ . We claim that the columns of the matrix  $E$  are related to the columns of the matrix  $H$ , as defined by (5.34), by the equations

$$e_{jk} = \begin{cases} h_{jk}, & k = 1, \\ \lambda_k \left[ h_{jk} + \sum_{i \in \mathbb{N}_{k-1}} \eta_{ki} h_{ji} \right], & k = 2, \dots, \mu, \end{cases} \quad j \in \mathbb{N}_\mu, \quad (5.48)$$

where, for  $k = 2, \dots, \mu$ ,

$$\eta_{ki} = (-1)^{i+k} \left[ \binom{2k-2}{k-i} - \binom{2k-2}{k-i-1} \right], \quad i \in \mathbb{N}_{k-1}, \quad (5.49)$$

and where

$$\lambda_k = \prod_{i \in \mathbb{N}_{2k-2}} \frac{1}{2\nu + i}. \quad (5.50)$$

To prove (5.48), we first note from (5.47) and (5.34) that

$$e_{j1} = h_{j1}, \quad j \in \mathbb{N}_\mu,$$

so that (5.48) holds for  $k = 1$ . Now, for  $k \in \{2, \dots, \mu\}$  and  $j \in \mathbb{N}_\mu$ , we find, by applying the recursive formula (5.1) repeatedly to the right-hand side of (5.47), and by recalling

the definition (5.34), that

$$\begin{aligned}
 \frac{e_{jk}}{\lambda_k} &= \sum_{i \in \mathbb{Z}_{2k-2}} (-1)^i \binom{2k-2}{i} [-\nu-k+1+i, \dots, \nu-k+1+i] f_j \\
 &\quad + \sum_{i \in \mathbb{Z}_{2k-2}} (-1)^i \binom{2k-2}{i} [-\nu-k+2+i, \dots, \nu-k+2+i] f_j \\
 &= \sum_{i \in \mathbb{Z}_{2k-2}} (-1)^i \binom{2k-2}{i} [-\nu-k+1+i, \dots, \nu-k+1+i] f_j \\
 &\quad + \sum_{i \in \mathbb{Z}_{2k-2}} (-1)^i \binom{2k-2}{2k-2-i} [-\nu+k-i, \dots, \nu+k-i] f_j \\
 &= \sum_{i \in \mathbb{Z}_{2k-2}} (-1)^i \binom{2k-2}{i} \left( [-\nu-k+1+i, \dots, \nu-k+1+i] f_j + [-\nu+k-i, \dots, \nu+k-i] f_j \right) \\
 &= h_{jk} + \sum_{i \in \mathbb{N}_{k-1}} (-1)^i \binom{2k-2}{i} \left( [-\nu-k+1+i, \dots, \nu-k+1+i] f_j + [-\nu+k-i, \dots, \nu+k-i] f_j \right) \\
 &\quad + \sum_{i \in \{k+\mathbb{Z}_{k-2}\}} (-1)^i \binom{2k-2}{i} \left( [-\nu-k+1+i, \dots, \nu-k+1+i] f_j + [-\nu+k-i, \dots, \nu+k-i] f_j \right) \\
 &= h_{jk} + \sum_{i \in \mathbb{N}_{k-1}} (-1)^{i+k} \binom{2k-2}{k-i} \left( [-\nu-i+1, \dots, \nu-i+1] f_j + [-\nu+i, \dots, \nu+i] f_j \right) \\
 &\quad + \sum_{i \in \mathbb{N}_{k-1}} (-1)^{i+k+1} \binom{2k-2}{i+k-1} \left( [-\nu+i, \dots, \nu+i] f_j + [-\nu-i+1, \dots, \nu-i+1] f_j \right) \\
 &= h_{jk} + \sum_{i \in \mathbb{N}_{k-1}} (-1)^{i+k} \binom{2k-2}{k-i} h_{ji} + \sum_{i \in \mathbb{N}_{k-1}} (-1)^{i+k+1} \binom{2k-2}{i+k-1} h_{ji} \\
 &= h_{jk} + \sum_{i \in \mathbb{N}_{k-1}} (-1)^{i+k} \left[ \binom{2k-2}{k-i} - \binom{2k-2}{k-i-1} \right] h_{ji},
 \end{aligned}$$

thereby proving (5.48) also for  $j \in \mathbb{N}_\mu$  and  $k \in \{2, \dots, \mu\}$ .

It follows from (5.48) that  $\det(E) = \left[ \prod_{k=2}^{\mu} \lambda_k \right] \det(H)$ . Since Proposition 5.5 gives  $\det(H) \neq 0$ , and since (5.50) implies that  $\lambda_k \neq 0$ ,  $k = 2, \dots, \mu$ , we therefore also have  $\det(E) \neq 0$ . ■

Combining the results of Theorem 5.4 and 5.6, we immediately deduce the following result.

**COROLLARY 5.7** *The Dubuc–Deslauriers expansion coefficient sequence  $\mathbf{t} = \{t_j : j \in \mathbb{Z}_\mu\}$*

of Theorem 5.4, can be obtained as the unique solution of the  $(\mu + 1) \times (\mu + 1)$  linear system (5.33).

We conclude that the computational algorithm for the mask symbol  $A \in \mathcal{A}_{\mu, \nu}$ , based on (1.3), (5.22), (2.65), (2.13) and (2.16), and in which we have to solve the  $(\mu + 1) \times (\mu + 1)$  linear system (5.33) to compute the coefficient sequence  $\{t_j : j \in \mathbb{Z}_\mu\}$  in (2.65), is more efficient than the one obtained from the algorithm based on (1.3), (5.22), (5.17) and the solving of the  $(2\mu + 2\nu) \times (2\mu + 2\nu)$  linear system (5.12) to compute, for each  $j \in \mathbb{J}_{\mu+\nu}$ , the coefficient sequence  $\{\beta_{ji} : i = -\mu + 1, \dots, \mu + 2\nu\}$  in (5.17).

We proceed to consider the special case  $\mu = 1$  of Theorem 5.3. Writing  $f = f_1$ , we deduce from Corollary 5.7 and Theorem 2.7 that the resulting mask symbol  $A$ , as defined by (1.3) and (5.22), is in  $\mathcal{A}_{1, \nu}$ , and is given by

$$A = t_0 D_\nu + t_1 D_{\nu+1}, \quad (5.51)$$

where  $\mathbf{t} = (t_0, t_1)$  is the unique solution of the  $2 \times 2$  linear system

$$B\mathbf{t} = \begin{bmatrix} 1 & 1 \\ \rho_\nu \left[ \frac{1}{2}, -\nu+1, \dots, \nu \right] f & \rho_{\nu+1} \left[ \frac{1}{2}, -\nu, \dots, \nu+1 \right] f \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (5.52)$$

so that

$$\begin{bmatrix} t_0 \\ t_1 \end{bmatrix} = \frac{1}{\det(B)} \begin{bmatrix} \rho_{\nu+1} \left[ \frac{1}{2}, -\nu, \dots, \nu+1 \right] f & -1 \\ -\rho_\nu \left[ \frac{1}{2}, -\nu+1, \dots, \nu \right] f & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (5.53)$$

Now observe, from the proof of Theorem 5.6, that, since  $\mu = 1$ , and using also (5.46) and (5.47), we have

$$\det(B) = -\det(C) = -\frac{1}{2} \rho_\nu \det(E) = -\frac{1}{2} \rho_\nu \left( \left[ \frac{1}{2}, -\nu, \dots, \nu \right] f + \left[ \frac{1}{2}, -\nu+1, \dots, \nu+1 \right] f \right). \quad (5.54)$$

It follows from (5.53), (5.54), and (5.30) that

$$t_0 = \frac{(2\nu + 1)^2}{2} \frac{\left[ \frac{1}{2}, -\nu, \dots, \nu+1 \right] f}{\left[ -\nu, \dots, \nu \right] f + \left[ -\nu+1, \dots, \nu+1 \right] f}, \quad (5.55)$$

$$t_1 = 2 \frac{\left[ \frac{1}{2}, -\nu+1, \dots, \nu \right] f}{\left[ -\nu, \dots, \nu \right] f + \left[ -\nu+1, \dots, \nu+1 \right] f}. \quad (5.56)$$

We will use the explicit formulation (5.51), (5.55), (5.56) of the mask symbol  $A$ , together with (2.13) and (2.16), for specific choices of  $f$  and  $\nu$  in the following section.

### 5.3 The case where $\{f_j : j \in \mathbb{J}_\mu\}$ are chosen as truncated powers

In this section we make a specific choice of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  in Theorem 5.2, and apply the analysis of Sections 5.1 and 5.2.

We will need the following facts from the theory of spline functions on arbitrary knot sequences (see e.g. [56, Chapter II, Section 2] or [57, Chapter 19 and 20]).

For a given  $m \in \mathbb{N}$ , and a partition  $\Pi = \{x_j : j \in \mathbb{Z}\}$  of  $\mathbb{R}$ , where  $x_j < x_{j+1}$ ,  $j \in \mathbb{Z}$ , we define the **spline space**  $S_m(\Pi)$  of order  $m$  and with respect to the partition  $\Pi$  as the set

$$S_m(\Pi) = \{s \in M(\mathbb{R}) : s|_{[x_j, x_{j+1})} \in \pi_{m-1}; s \in C^{m-2}(\mathbb{R})\}.$$

Then, with the definition

$$N_{m,j}(x) = (-1)^m (x_{j+m} - x_j) [x_j, \dots, x_{j+m}] (x - \cdot)_+^{m-1}, \quad x \in \mathbb{R}, \quad m \in \mathbb{N}, \quad (5.57)$$

we have, for  $m \in \mathbb{N}$ , that the following properties are satisfied:

$$\left. \begin{aligned} N_{m,j} &= (x_{j+m} - x_j) \sum_{k \in \{j + \mathbb{Z}_m\}} b_{j,k} (\cdot - x_k)_+^{m-1}, \\ \text{where} \quad b_{j,k} &= \prod_{\ell \in \{j + \mathbb{Z}_m\}} \frac{1}{(x_\ell - x_k)}, \quad k \in \{j + \mathbb{Z}_m\}, \end{aligned} \right\} \quad j \in \mathbb{Z}; \quad (5.58)$$

$$N_{m,j} \in S_m(\Pi), \quad j \in \mathbb{Z};$$

$$N_{m,j}(x) = 0, \quad x \notin (x_j, x_{j+m}), \quad j \in \mathbb{Z}, \quad (m \geq 2); \quad (5.59)$$

$$N_{m,j}(x) > 0, \quad x \in (x_j, x_{j+m}), \quad j \in \mathbb{Z}; \quad (5.60)$$

$$\sum_j N_{m,j}(x) = 1, \quad x \in \mathbb{R};$$

$$N_{m,j} = \frac{\cdot - x_j}{x_{j+m-1} - x_j} N_{m-1,j} + \frac{x_{j-m} - \cdot}{x_{j-m} - x_{j+1}} N_{m-1,j+1}, \quad j \in \mathbb{Z}, \quad (m \geq 2). \quad (5.61)$$

Moreover, the sequence  $\{N_{m,j} : j \in \mathbb{Z}\}$  is a basis for  $S_m(\Pi)$  in the sense that, for every  $s \in S_m(\Pi)$ , there exists a unique sequence  $c \in M(\mathbb{Z})$  such that  $s = \sum_j c_j N_{m,j}$ . If, for a



given integer  $\ell \in \mathbb{Z}$ , we have  $x_j = \ell + j$ ,  $j \in \mathbb{Z}$ , so that  $\Pi = \mathbb{Z}$ , then

$$N_{m,j} = N_m(\cdot - \ell - j), \quad j \in \mathbb{Z}, \quad (5.62)$$

with  $N_m$  denoting the cardinal B-spline of order  $m$ , as introduced in Section 1.2. The sequence  $\{N_{m,j} : j \in \mathbb{Z}\}$  is called the sequence of  $m$ -th order B-splines with respect to the knot sequence  $\Pi$ . Note in particular from (5.57) that, for a fixed  $j \in \mathbb{Z}$ , the B-spline  $N_{m,j}$  depends only on the finite knot sequence  $\{x_j, \dots, x_{j+m}\}$ . Also, from (5.57) and (5.58), we deduce, for  $m \geq 3$  and  $k \in \mathbb{N}_{m-2}$ , the differentiation formulas

$$N_{m,j}^{(k)}(x) = (-1)^m \frac{(m-1)!}{(m-1-k)!} (x_{j+m} - x_j) [x_j, \dots, x_{j+m}] (x - \cdot)_+^{m-1-k}, \quad x \in \mathbb{R}, \quad (5.63)$$

and

$$N_{m,j}^{(k)} = \frac{(m-1)!}{(m-1-k)!} (x_{j+m} - x_j) \sum_{k \in \{j+\mathbb{Z}_m\}} b_{j,k} (\cdot - x_k)_+^{m-1-k}, \quad (5.64)$$

with  $b_{j,k}$  as in the second equation in (5.58).

As an example of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$ , which, in the context of Theorem 5.3, yields the mask symbol  $\mathbf{A} \in \mathcal{A}_{\mu,\nu}$  by means of (1.3) and (5.22), we now choose, as was done in [22], the truncated powers

$$f_j = (\xi_j - \cdot)_+^{2\nu-1}, \quad j \in \mathbb{N}_\mu, \quad (5.65)$$

$$f_j = f_{1-j}(1 - \cdot), \quad j = -\mu + 1, \dots, 0, \quad (5.66)$$

where

$$\left. \begin{aligned} \xi_j &\in \left(\frac{1}{2}, \nu + j\right), \quad j \in \mathbb{N}_\mu, \\ \xi_j &< \xi_{j+1}, \quad j \in \mathbb{N}_{\mu-1}, \quad (\text{if } \mu \geq 2). \end{aligned} \right\} \quad (5.67)$$

The following result then shows that this choice is indeed an admissible one.

**PROPOSITION 5.8** *For  $\mu, \nu \in \mathbb{N}$ , with  $\mu \leq \nu$ , if the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  is defined by (5.65), (5.66), and (5.67), then the matrix  $\mathbf{X}$  defined by (5.6) is invertible.*

**Proof.** First observe from (5.65), (5.66), (5.67), (5.57) and (5.62), with  $\ell = -\nu$ , that if we define the  $2\mu \times 2\mu$  matrix  $Y$  by

$$(Y)_{ij} = \begin{cases} N_{2\nu}(\xi_j + \nu - i), & j \in \mathbb{N}_\mu, \\ N_{2\nu}(1 - \xi_{1-j} + \nu - i), & j = -\mu + 1, \dots, 0, \end{cases} \quad i \in \mathbb{J}_\mu, \quad (5.68)$$

then, from (5.6), we have  $\det(X) = (-1)^\nu (2\nu)^{-\mu^2} \det(Y)$ . Hence our proof will be complete if we can show that  $\det(Y) \neq 0$ . According to the Schoenberg–Whitney Theorem [57, Theorem 19.4], we know that the matrix  $Y$  defined by (5.68) satisfies  $\det(Y) \neq 0$  if and only if

$$\left. \begin{aligned} N_{2\nu}(\xi_j + \nu - j) &> 0, & j \in \mathbb{N}_\mu, \\ N_{2\nu}(1 - \xi_{1-j} + \nu - j) &> 0, & j = -\mu + 1, \dots, 0. \end{aligned} \right\} \quad (5.69)$$

But, from (5.67) and the fact that  $\mu \leq \nu$ , we have

$$\frac{1}{2} \leq \frac{1}{2} + \nu - \mu < \xi_j + \nu - j < (\nu + j) + \nu - j = 2\nu, \quad j \in \mathbb{N}_\mu,$$

and, for  $j = -\mu + 1, \dots, 0$ ,

$$0 = 1 - (\nu + (1 - j)) + \nu - j < 1 - \xi_{1-j} + \nu - j < \frac{1}{2} + \nu - j \leq \nu + \mu - \frac{1}{2} \leq 2\nu - \frac{1}{2}.$$

Thus

$$\left. \begin{aligned} \xi_j + \nu - j &\in (0, 2\nu), & j \in \mathbb{N}_\mu, \\ 1 - \xi_{1-j} + \nu - j &\in (0, 2\nu), & j = -\mu + 1, \dots, 0, \end{aligned} \right\}$$

which, together with (1.18), shows that the Schoenberg–Whitney condition (5.69) is indeed satisfied. Hence  $\det(Y) \neq 0$ . ■

Combining the results of Proposition 5.8 and Theorem 5.3, we deduce the following.

**COROLLARY 5.9** *For  $\mu, \nu \in \mathbb{N}$ , with  $\mu \leq \nu$ , if the mask symbol  $A$  defined by (1.3) and (5.22) is based on the choice (5.65), (5.66), (5.67) of the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  in Theorem 5.2, then  $A \in \mathcal{A}_{\mu, \nu}$ .*

We can now use Corollary 5.7 to calculate the coefficients  $\{t_j : j \in \mathbb{Z}\}$  of the Dubuc–Deslauriers expansion (2.65) of the mask symbol  $A \in \mathcal{A}_{\mu, \nu}$  of Corollary 5.9. To this

end, we first use the definition (5.57) to define the sequence

$$\tilde{N}_{2j} = N_{2j, -j+1} \in S_{2j}(\tilde{\Pi}), \quad j \in \mathbb{N}, \quad (5.70)$$

where the partition  $\tilde{\Pi} = \{\tilde{x}_k : k \in \mathbb{Z}\}$  of  $\mathbb{R}$  is defined by

$$\tilde{x}_k = \begin{cases} k, & k \leq 0, \\ \frac{1}{2}, & k = 1, \\ k - 1, & k \geq 2. \end{cases} \quad (5.71)$$

Note from (5.70), (5.59) and (5.60) that

$$\tilde{N}_{2j}(x) = 0, \quad x \notin (-j+1, j), \quad j \in \mathbb{N}, \quad (5.72)$$

$$\tilde{N}_{2j}(x) > 0, \quad x \in (-j+1, j), \quad j \in \mathbb{N}. \quad (5.73)$$

Also, the B-spline  $\tilde{N}_{2j}$  is symmetric about  $\frac{1}{2}$ , i.e.

$$\tilde{N}_{2j}(1 - \cdot) = \tilde{N}_{2j}, \quad j \in \mathbb{N}, \quad (5.74)$$

as is proved by using (5.57) and the fact that a divided difference is a symmetric function of its arguments.

It then follows from (5.32), (5.70), and (5.63) that, for the mask symbol  $A \in \mathcal{A}_{\mu, \nu}$  of Corollary 5.9, the coefficient sequence  $\mathbf{t} = \{t_j : j \in \mathbb{Z}_\mu\}$  in the Dubuc–Deslauriers expansion (2.65) is the unique solution  $\mathbf{t}$  of the  $(\mu+1) \times (\mu+1)$  linear system (5.33), where, using again also the fact that a divided difference is a symmetric function of its arguments, the matrix  $B$  is given by

$$(B)_{ij} = \begin{cases} 1, & i = 0, \\ \rho_{\nu+j} \frac{(2\nu-1)!}{(2\nu+2j-1)(2\nu+2j-1)!} \tilde{N}_{2\nu+2j}^{(2j)}(\xi_i), & i \in \mathbb{N}_\mu, \end{cases} \quad j \in \mathbb{Z}_\mu,$$

or, equivalently, using also (5.29),

$$(B)_{ij} = \begin{cases} 1, & i = 0, \\ \frac{(-1)^{\nu+j}(2\nu-1)!}{2^{4\nu+4j-2}} \binom{2\nu+2j-2}{\nu+j-1} \tilde{N}_{2\nu+2j}^{(2j)}(\xi_i), & i \in \mathbb{N}_\mu, \end{cases} \quad j \in \mathbb{Z}_\mu, \quad (5.75)$$

with the convention that  $\binom{0}{0} = 1$ . We are indebted to Charles Micchelli for pointing out to us that, in the case where the sequence  $\{f_j : j \in \mathbb{J}_\mu\}$  is given by (5.65), (5.66), the second line of (5.32) can be expressed in terms of B-spline derivatives, as in (5.75), thereby eventually leading to the useful existence and convergence criterium in Theorem 5.10 below, as based on (5.82), together with (5.77), (5.78), (5.51) and (2.35).

Hence, for the mask symbol  $A \in \mathcal{A}_{\mu,\nu}$ , of Corollary 5.9, the coefficient sequence  $\{t_j : j \in \mathbb{Z}_\mu\}$  in the Dubuc–Deslauriers expansion (2.65) of  $A$  can be computed by solving the linear system (5.33), with  $B$ ,  $\mathbf{t}$  and  $\mathbf{b}$  as given by (5.75) and (5.31).

The existence of a corresponding refinable function  $\phi$ , and the convergence of the associated subdivision scheme  $S_a$  can then be studied by, for example, investigating whether the sufficient conditions of either Theorem 2.2 or Theorem 4.8 are satisfied by  $A$ .

We shall consider here the case  $\mu = 1$ ,  $\nu \in \mathbb{N}$ , of Corollary 5.9. In this case  $A$  has the form (5.51), where, writing  $f = f_1$  and  $\xi = \xi_1$ , we find from (5.55), (5.56), (5.65), (5.66), (5.57) and (5.63), that with

$$\xi = \xi_1 \in \left(\frac{1}{2}, \nu + 1\right), \quad (5.76)$$

the formulas

$$t_0 = t_0(\xi) = \frac{1}{2} \frac{\tilde{N}_{2\nu+2}''(\xi)}{N_{2\nu}(\xi + \nu) + N_{2\nu}(\xi + \nu - 1)}, \quad (5.77)$$

$$t_1 = t_1(\xi) = \frac{4\nu}{2\nu - 1} \frac{\tilde{N}_{2\nu}(\xi)}{N_{2\nu}(\xi + \nu) + N_{2\nu}(\xi + \nu - 1)}, \quad (5.78)$$

are satisfied.

For  $\xi \in \left(\frac{1}{2}, \nu + 1\right)$ , we now observe from (5.51), (4.1) and Proposition 4.2 (with  $n = \nu + 1$ ), together with (5.78), that, for  $\nu \in \mathbb{N}$ , the mask  $A \in \mathcal{A}_{1,\nu}$  in Corollary 5.9 satisfies the positivity condition (2.23) of Theorem 2.2 if

$$-\frac{1}{2}(2\nu - 1) \leq \frac{\tilde{N}_{2\nu}(\xi)}{N_{2\nu}(\xi + \nu) + N_{2\nu}(\xi + \nu - 1)} \leq \frac{2\nu - 1}{4\nu}, \quad (5.79)$$

whereas, if  $\tilde{N}_{2\nu+2}''(\xi) < 0$ , then  $A$  has two zeros on the unit circle in  $\mathbb{C}$ .

Since  $-\frac{1}{2}(2\nu - 1) \leq 0$ ,  $\nu \in \mathbb{N}$ , we deduce from (5.70), (5.60) and (1.18) that the left-hand inequality in (5.79) holds for all  $\xi \in (\frac{1}{2}, \nu + 1)$ . Also, from (5.72), we see that the right-hand side inequality in (5.79) holds for  $\xi \in [\nu, \nu + 1)$ , with, from (5.78),

$$t_1(\xi) = 0, \quad \xi \in [\nu, \nu + 1). \quad (5.80)$$

Next, since Theorem 2.7 gives  $t_0 = 1 - t_1$ , we deduce from (5.79) and (5.77), and using (1.18) once again, that the right-hand inequality in (5.79) holds if and only if  $\xi \in (\frac{1}{2}, \nu + 1)$  is such that

$$\tilde{N}_{2\nu+2}''(\xi) \geq 0. \quad (5.81)$$

But, from [56, Chapter II, Theorem 2.5], together with the definition (5.70), we conclude that there exists exactly one point  $\xi_\nu^* \in (\frac{1}{2}, \nu + 1)$  such that

$$\tilde{N}_{2\nu+2}''(\xi) \begin{cases} < 0, & \frac{1}{2} < \xi < \xi_\nu^*, \\ = 0, & \xi = \xi_\nu^*, \\ > 0, & \xi_\nu^* < \xi < \nu + 1. \end{cases} \quad (5.82)$$

Next, we use (5.77) and (5.78), together with the fact that  $t_0 = 1 - t_1$ , and since also  $\tilde{N}_{2\nu}(\nu) = 0$  from (5.72), to obtain

$$\tilde{N}_{2\nu+2}''(\nu) = 2 N_{2\nu}(2\nu - 1) > 0, \quad \nu \in \mathbb{N}, \quad (5.83)$$

by virtue also of (1.17) and (1.18). It then follows from (5.83) and (5.82) that

$$\xi_\nu^* \in (\frac{1}{2}, \nu). \quad (5.84)$$

In particular, combining the results (5.77) – (5.84), and recalling also from (2.13) and (2.16) that  $\deg(D_n) = 2n - 1$ ,  $n \in \mathbb{N}$ , we have therefore established the following result.

**THEOREM 5.10** *For  $\nu \in \mathbb{N}$ , let  $A \in \mathcal{A}_{1,\nu}$  denote the mask symbol obtained by setting  $\mu = 1$  in Corollary 5.9. With  $\xi_\nu^*$  denoting the unique point in  $(\frac{1}{2}, \nu)$  such that  $\tilde{N}_{2\nu+2}''(\xi_\nu^*) = 0$ ,*

we have that, if the condition  $\xi \in [\xi_\nu^*, \nu + 1)$  holds, then  $\mathbf{A}$  satisfies the conditions of Theorem 2.2 with

$$n = \begin{cases} \nu + 1, & \xi_\nu^* \leq \xi < \nu, \\ \nu, & \nu \leq \xi < \nu + 1, \end{cases} \quad (5.85)$$

whereas, if  $\xi \in (\frac{1}{2}, \xi_\nu^*)$ , then  $\mathbf{A}$  has two zeros on the unit circle in  $\mathbb{C}$ . Moreover,

$$\mathbf{A} = \begin{cases} \mathbf{D}_{\nu+1}, & \xi = \xi_\nu^*, \\ \mathbf{D}_\nu, & \xi \in [\nu, \nu + 1). \end{cases}$$

**Examples.** We proceed to consider the special cases  $\nu = 1$  and  $\nu = 2$  of Theorem 5.10.

(a) First, setting  $\nu = 1$  in Theorem 5.10, we seek the unique point  $\xi_1^* \in (\frac{1}{2}, 1)$  which is such that

$$\tilde{\mathbf{N}}_4''(\xi_1^*) = 0. \quad (5.86)$$

But, since  $t_0 + t_1 = 1$ , we deduce from (5.77) and (5.78) that (5.86) holds if and only if  $\xi_1^* \in (\frac{1}{2}, 1)$  is such that

$$\frac{N_2(2\xi_1^*)}{N_2(\xi_1^* + 1) + N_2(\xi_1^*)} = \frac{1}{4}, \quad (5.87)$$

since also, from (5.70), we have that  $\tilde{\mathbf{N}}_2 = N_2(2\cdot)$ . It follows from (5.87) and (1.23) that

$$\frac{2 - 2\xi_1^*}{[2 - (\xi_1^* + 1)] + \xi_1^*} = \frac{1}{4},$$

and thus  $\xi_1^* = \frac{7}{8}$ .

Hence, according to Theorem 5.10, the condition  $\xi \in [\frac{7}{8}, 2)$  guarantees that the mask symbol  $\mathbf{A} \in \mathcal{A}_{1,1}$  of that theorem satisfies the conditions of Theorem 2.2 with

$$n = \begin{cases} 2, & \frac{7}{8} \leq \xi < 1, \\ 1, & 1 \leq \xi < 2. \end{cases}$$

Moreover,  $\mathbf{A} = \mathbf{D}_2$  if  $\xi = \frac{7}{8}$ , and  $\mathbf{A} = \mathbf{D}_1$  if  $\xi \in [1, 2)$ .

Now recall from Section 4.3.1(a), and according to (4.58), (5.51) and (5.78) and the fact that  $\tilde{\mathbf{N}}_2 = N_2(2\cdot)$ , that our mask  $\mathbf{A}$  considered here satisfies the conditions of Theorem 4.8 if and only if  $\xi \in (\frac{1}{2}, 2)$  is such that  $|t_1(\xi)| < 2$ , i.e.

$$-\frac{1}{2} < \frac{N_2(2\xi)}{N_2(\xi + 1) + N_2(\xi)} < \frac{1}{2}. \quad (5.88)$$

From (1.18), we see that the left-hand inequality of (5.88) holds for all  $\xi \in (\frac{1}{2}, 2)$ , whereas, if  $\xi \in [1, 2)$ , then (1.17) yields  $N_2(2\xi) = 0$ , so that the right-hand inequality of (5.88) holds. If  $\xi \in (\frac{1}{2}, 1)$ , then (1.23) shows that

$$\frac{N_2(2\xi)}{N_2(\xi+1) + N_2(\xi)} = \frac{2-2\xi}{[2-(\xi+1)] + \xi} = 2-2\xi,$$

i.e. the right-hand inequality in (5.88) holds if and only if  $\xi \in (\frac{3}{4}, 1)$ .

Hence  $|t(\xi)| = |t_1(\xi)| < 2$  if and only if  $\xi \in (\frac{3}{4}, 2)$ , so that our Theorem 4.8 actually extends the existence and convergence  $\xi$ -interval from  $\xi \in [\frac{7}{8}, 2)$  to  $\xi \in (\frac{3}{4}, 2)$  and where the interval  $\xi \in (\frac{3}{4}, \frac{7}{8})$ , according to Theorem 5.10, produces mask symbols  $A$  with zeros on the unit circle in  $\mathbb{C}$ .

For  $\xi = \frac{13}{16} \in (\frac{3}{4}, \frac{7}{8})$ , it follows from (5.78) that

$$t_1\left(\frac{13}{16}\right) = 4 \frac{N_2\left(\frac{13}{8}\right)}{N_2\left(\frac{29}{16}\right) + N_2\left(\frac{13}{16}\right)} = \frac{3}{2},$$

and thus  $t_0\left(\frac{13}{16}\right) = 1 - t_1\left(\frac{13}{16}\right) = -\frac{1}{2}$ , which, together with (5.51) and (4.1) with  $n = 2$ , yields once again the mask symbol (4.13), as given by

$$A(z) = A\left(\frac{3}{2}|z\right) = -\frac{3}{32}z^{-3} + \frac{19}{32}z^{-1} + 1 + \frac{19}{32}z - \frac{3}{32}z^3, \quad z \in \mathbb{C} \setminus \{0\}, \quad (5.89)$$

and with corresponding refinable function  $\phi$  plotted in Figure 4.1. Here we illustrate the associated convergent subdivision scheme in Figure 5.1

(b) Next, we set  $v = 2$  in Theorem 5.10, and proceed to seek the unique point  $\xi_2^* \in (\frac{1}{2}, 2)$  which is such that  $\tilde{N}_6''(\xi_2^*) = 0$ , i.e., from (5.74), we shall seek the unique point  $\eta_2^* \in (-1, \frac{1}{2})$  which is such that  $\tilde{N}_6''(\eta_2^*) = 0$ , and then set  $\xi_2^* = 1 - \eta_2^*$ . From (5.70), we therefore need to find the point  $\eta_2^* \in (-1, \frac{1}{2})$  such that  $N_{6,-2}''(\eta_2^*) = 0$ , which, from the differentiation formula (5.64), and since the partition  $\Pi = \tilde{\Pi} = \{\tilde{x}_k : k \in \mathbb{Z}\}$  is defined by (5.71), is equivalent to the equation

$$\sum_{k=-2}^0 \left[ \prod_{\ell \neq k}^4 \frac{1}{\tilde{x}_\ell - \tilde{x}_k} \right] (\eta_2^* - \tilde{x}_k)_+^3 = 0, \quad (5.90)$$

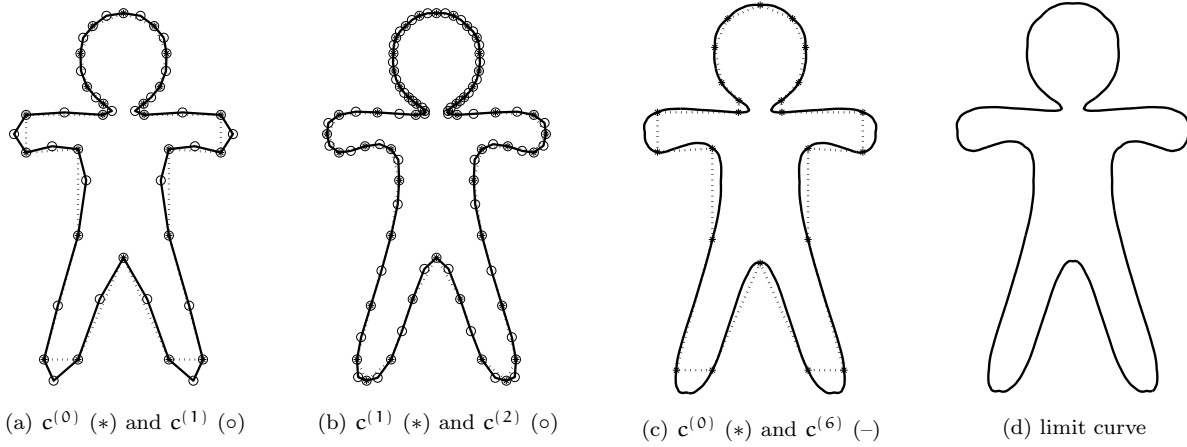


Figure 5.1: Illustration of subdivision with mask symbol (5.89)

where

$$\tilde{x}_{-2} = -2, \quad \tilde{x}_{-1} = -1, \quad \tilde{x}_0 = 0, \quad \tilde{x}_1 = \frac{1}{2}, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 2, \quad \tilde{x}_4 = 3. \quad (5.91)$$

To solve (5.90), we first consider the possibility  $\eta_2^* \in (-1, 0)$ . But then the equation (5.90) is given by

$$\sum_{k=-2}^{-1} \left[ \prod_{\ell \neq k}^4 \frac{1}{\tilde{x}_\ell - \tilde{x}_k} \right] (\eta_2^* - \tilde{x}_k)^3 = 0,$$

or, equivalently,

$$\frac{1}{300} (\eta_2^* + 2)^3 - \frac{1}{36} (\eta_2^* + 1)^3 = 0,$$

so that

$$\eta_2^* = \frac{\sqrt[3]{\frac{25}{3}} - 2}{1 - \sqrt[3]{\frac{25}{3}}} \approx -0.0267,$$

which is indeed in the correct interval  $(-1, 0)$ . It follows that

$$\xi_2^* = 1 - \eta_2^* = \frac{3 - 2\sqrt[3]{\frac{25}{3}}}{1 - \sqrt[3]{\frac{25}{3}}} \approx 1.0267$$

is the unique point in  $(\frac{1}{2}, 2)$  such that  $\tilde{N}_6''(\xi_2^*) = 0$ . Hence, from Theorem 5.10, we deduce that our mask symbol  $\mathbf{A} \in \mathcal{A}_{1,2}$  satisfies the conditions of Theorem 2.2 if  $\xi \in [\xi_2^*, 3) \approx [1.0267, 3)$ , with

$$n = \begin{cases} 3, & \xi_2^* \leq \xi < 2, \\ 2, & 2 \leq \xi < 3, \end{cases}$$



and where  $\mathbf{A} = \mathbf{D}_3$  if  $\xi = \xi_2^*$ , and  $\mathbf{A} = \mathbf{D}_2$  if  $\xi \in [2, 3)$ .

If  $\xi \in (\frac{1}{2}, \xi_2^*)$ , it follows from (5.82), (5.77), (5.78), and the condition  $\mathbf{t}_0(\xi) + \mathbf{t}_1(\xi) = 1$ , that  $\mathbf{t}(\xi) = \mathbf{t}_1(\xi) > 1$ , and thus, from (5.51) and Proposition 4.2 (with  $\mathbf{n} = 3$ ), we deduce that  $\mathbf{A}$  then has two zeros on the unit circle in  $\mathbb{C}$ . According to example (b) in Section 4.3.1, the conditions of Theorem 4.8 are satisfied by  $\mathbf{A}$ , as given by (5.51), (5.77), (5.78) and (5.70), if and only if

$$-4 < \frac{N_{4,-1}(\xi)}{N_4(\xi+2) + N_4(\xi+1)} < 2. \quad (5.92)$$

From (1.18) and (5.74), we see that it suffices to find  $\eta \in (-1, \frac{1}{2})$  such that the inequality

$$N_{4,-1}(\eta) - 2 [N_4(\eta+2) + N_4(\eta+1)] < 0 \quad (5.93)$$

is satisfied, having also noted from (1.17) and (5.72) that the inequality (5.92) holds, with the expression in the middle of (5.92) equalling zero if  $\eta \in (-2, -1]$ .

Using (5.58) and (1.13), we find that (5.93) is equivalent to the inequality

$$\sum_{k=-1}^0 \left[ \prod_{\ell \neq k}^3 \frac{1}{\tilde{x}_\ell - \tilde{x}_k} \right] (\eta - \tilde{x}_k)_+^3 - 2 \left[ \sum_{k \in \mathbb{Z}_1} (-1)^k \binom{4}{k} (\eta + 2 - k)_+^3 + (\eta + 1)_+^3 \right] < 0, \quad (5.94)$$

with the knot sequence  $\{\tilde{x}_k : k \in \{-2 + \mathbb{Z}_6\}\}$  as given by (5.91), so that (5.94) is equivalent to

$$-\eta_+^3 + 7(\eta + 1)_+^3 - 2(\eta + 2)_+^3 < 0. \quad (5.95)$$

A routine calculus procedure now shows that the inequality (5.95) holds for all  $\eta \in (-1, \frac{1}{2})$ , thereby proving that (5.92) holds for all  $\xi \in (\frac{1}{2}, 3)$ .

Our result of Theorem 4.8 therefore extends the existence and convergence interval  $\xi \in [\xi_2^*, 3) \approx [1.0267, 3)$ , as obtained from Theorems 5.10 and 2.2, to  $\xi \in (\frac{1}{2}, 3)$ , which is the entire range of  $\xi$  under consideration in this example.

For example, setting  $\xi = \frac{2}{3} \in (\frac{1}{2}, 3)$  and  $\mathbf{v} = 2$  in (5.78), yields

$$\mathbf{t}_1\left(\frac{2}{3}\right) = \frac{8}{3} \frac{N_{4,-1}\left(\frac{2}{3}\right)}{N_4\left(\frac{8}{3}\right) + N_4\left(\frac{5}{3}\right)} = \frac{880}{459},$$

and since  $t_0\left(\frac{2}{3}\right) = 1 - t_1\left(\frac{2}{3}\right) = -\frac{421}{459}$ , we find from (5.51), (2.18) and (2.19) that

$$A(z) = \frac{1}{2448} \left( 55z^{-5} - 318z^{-3} + 1487z^{-1} + 2448 + 1487z - 318z^3 + 55z^5 \right), \quad z \in \mathbb{C} \setminus \{0\}. \quad (5.96)$$

The corresponding refinable function  $\phi$  and the associated convergent subdivision scheme are illustrated in Figures 5.2 and 5.3, respectively.

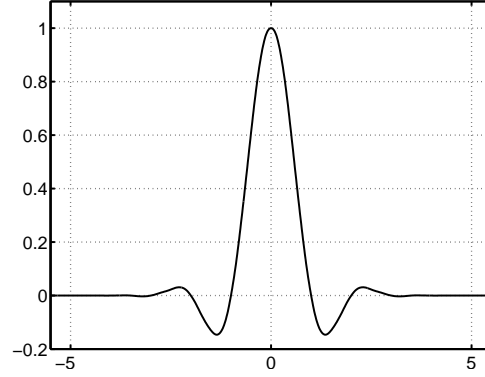


Figure 5.2: Refinable function with mask symbol (5.96)

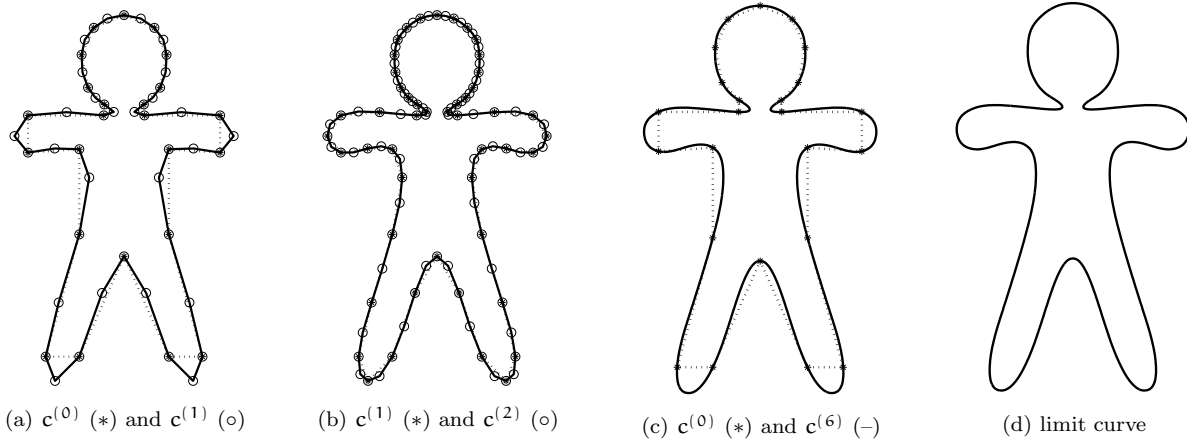
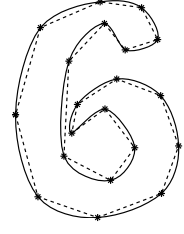


Figure 5.3: Illustration of subdivision with mask symbol (5.96)

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# Dubuc–Deslauriers subdivision for finite sequences

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The subdivision schemes of the previous chapters are all based on the availability of a bi-infinite initial sequence  $\mathbf{c}$  in (1.4), as is the case, for example, if  $\mathbf{c}$  is a periodic sequence. However, in applications one is often confronted with finite initial sequences, in which case modifications of the subdivision scheme are required to accommodate the boundaries. Following [21], we present a method of adapting the Dubuc–Deslauriers subdivision scheme  $S_{\mathbf{d}_n}$ , as introduced in Section 2.2, to be applicable when the initial sequence  $\mathbf{c}$  in (1.4) is finite.

For the remaining part of this thesis, we shall use, for  $\mathbf{n} \in \mathbb{N}$ , the simplified notation

$$\phi = \phi_n^D; \quad S = S_{\mathbf{d}_n}; \quad \mathbf{d}_j = \mathbf{d}_{\mathbf{n},j}, \quad j \in \mathbb{J}_n; \quad \ell_j = \ell_{\mathbf{n},j}, \quad j \in \mathbb{J}_n. \quad (6.1)$$

We base our construction on a multi-scale sequence of fundamental interpolants  $\{\phi_j^r\}$  defined on a bounded interval. Away from the boundaries the integer shifts of the original functions  $\phi$  suffice, while the adjustments in the proximity of the boundaries preserve the polynomial reproduction property (2.63) of  $\phi$ . The resulting adapted interpolatory subdivision scheme, which corresponds to Dubuc–Deslauriers subdivision away from the boundaries, then also preserves the polynomial filling property (2.14) of the subdivision scheme  $S$ . In fact, most results derived in this paper depend on these polynomial filling or polynomial reproduction properties.

## 6.1 Construction of a modified scheme

We first adapt the Dubuc–Deslauriers refinable functions to accommodate the boundaries of an interval and then use these adapted refinable functions construct a corresponding subdivision scheme for finite sequences.

For  $n \in \mathbb{N}$ , we derive the specific properties of these refinable functions from the Dubuc–Deslauriers refinable function  $\phi = \phi_n^D$  of Section 2.3, with appropriate modifications near the boundaries.

With  $n \in \mathbb{N}$ , as in Section 2.2, let  $L$  be a positive integer with  $L \geq 4n - 2$ , and let  $r$  be a nonnegative integer. On the basis of Theorems 2.2 and 2.6, and the subsequent equation (2.61), we seek to construct a sequence  $\{\phi_j^r\} = \{\phi_j^r : j \in \mathbb{Z}_{2^r L}, r \in \mathbb{Z}_+\}$  such that, for each fixed  $r \in \mathbb{Z}_+$ ,

$$\phi_j^r \in C[0, 2^r L], \quad j \in \mathbb{Z}_{2^r L}; \quad (6.2)$$

$$\phi_j^r(x) = 0, \quad x \notin \begin{cases} [0, 2n - 1 + j), & j \in \mathbb{Z}_{2n-1}, \\ (-2n + 1 + j, 2n - 1 + j), & j = 2n, \dots, 2^r L - 2n, \\ (-2n + 1 + j, 2^r L], & j = 2^r L - 2n + 1, \dots, 2^r L; \end{cases} \quad (6.3)$$

$$\phi_j^r(k) = \delta_{j,k}, \quad j, k \in \mathbb{Z}_{2^r L}; \quad (6.4)$$

$$\sum_{j \in \mathbb{Z}_{2^r L}} p(j) \phi_j^r(x) = p(x), \quad x \in [0, 2^r L], \quad p \in \pi_{2n-1}; \quad (6.5)$$

$$\phi_j^r(x) = \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x), \quad x \in [0, 2^r L], \quad j \in \mathbb{Z}_{2^r L}. \quad (6.6)$$

Denoting the linear space of finite real sequences  $c = \{c_j : j \in \mathbb{Z}_{2^r L}\}$  by  $M_r$ , i.e.,  $M_r = \mathbb{R}^{2^r L+1}$ , we define the subdivision operator sequence  $\{S_r : r \in \mathbb{Z}_+\}$  for  $S_r : M_r \rightarrow M_{r+1}$ , by

$$(S_r c)_j = \sum_{k \in \mathbb{Z}_{2^r L}} \phi_k^r\left(\frac{j}{2}\right) c_k, \quad j \in \mathbb{Z}_{2^{r+1}L}, \quad r \in \mathbb{Z}_+. \quad (6.7)$$

The corresponding subdivision scheme is defined by

$$\mathbf{c}^{(0)} = \mathbf{c} \in \mathcal{M}_0, \quad \mathbf{c}^{(r)} = \mathbf{S}_{r-1} \mathbf{c}^{(r-1)}, \quad r \in \mathbb{N}, \quad (6.8)$$

or, equivalently,

$$\mathbf{c}^{(0)} = \mathbf{c}, \quad \mathbf{c}^{(r)} = \mathbf{S}_{r-1}(\dots(\mathbf{S}_1(\mathbf{S}_0 \mathbf{c}) \dots)), \quad r \in \mathbb{N}. \quad (6.9)$$

Observe that (6.4), (6.7) and (6.8) imply

$$\mathbf{c}_{2^r j}^{(r)} = \mathbf{c}_j^{(r-1)}, \quad j \in \mathbb{Z}_{2^{r-1}L}, \quad r \in \mathbb{N}, \quad (6.10)$$

i.e. the subdivision scheme (6.7), (6.8) is interpolatory, whereas (6.5) implies

$$\sum_{k \in \mathbb{Z}_{2^r L}} \Phi_k^r \left( \frac{j}{2} \right) p(k) = p \left( \frac{j}{2} \right), \quad j \in \mathbb{Z}_{2^{r+1}L}, \quad p \in \pi_{2n-1}, \quad (6.11)$$

according to which the subdivision scheme (6.7), (6.8) has the  $(2n-1)$ -th degree polynomial filling property. In (6.7) – (6.11) we have obtained the analogues of (1.2), (1.4), (1.5), (2.2) and (2.14), respectively.

To find a sequence  $\{\Phi_j^r\}$  satisfying (6.2) – (6.6), we observe from (2.31) and (2.63) that,

$$\sum_{j \in \mathbb{Z}_{2^r L}} p(j) \Phi(x-j) = \sum_j p(j) \Phi(x-j) = p(x), \quad p \in \pi_{2n-1}, \quad x \in [2n-2, 2^r L - 2n+2]. \quad (6.12)$$

Thus, the sequence  $\{\Phi(\cdot - j)\}$  provides suitable refinable functions away from the boundaries, however closer to the boundaries of the interval some modifications are necessary. These boundary modifications can again be based on property (2.63). Using arguments similar to the ones which led to the construction of the mask (2.12), we define, for each fixed  $r \in \mathbb{Z}_+$ , the sequence  $\{\Phi_j^r\}$  on the interval  $[0, 2^r L]$  by

$$\Phi_j^r(x) = \begin{cases} \Phi(x-j) + \sum_{k=-2n+2}^{-1} \ell_{j-n+1}(k-n+1) \Phi(x-k), & j \in \mathbb{Z}_{2n-1}, & (6.13a) \\ \Phi(x-j), & j = 2n, \dots, 2^r L - 2n, & (6.13b) \\ \Phi_{2^r L-j}^r(2^r L - x), & j = 2^r L - 2n + 1, \dots, 2^r L, & (6.13c) \end{cases}$$

where the Lagrange fundamental polynomial polynomials  $\{\ell_k : k \in \mathbb{J}_n\} \subset \pi_{2n-1}$  are given by (2.8). Examples of these modified functions are plotted in Figure 6.1.

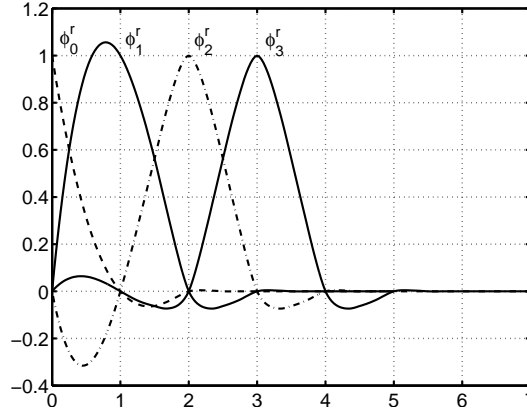


Figure 6.1: Illustration of definition (6.13) with  $n = 2$ .

Using

$$\phi\left(n + \frac{1}{2} + k\right) = 0, \quad k \in \mathbb{Z}_+, \quad (6.14)$$

and

$$\phi\left(-n - \frac{1}{2} - k\right) = 0, \quad k \in \mathbb{Z}_+, \quad (6.15)$$

which follow from (2.57) and (2.62) with  $m = 0$ , we prove the following useful properties of the sequence  $\{\phi_j^{(r)}\}$ .

**PROPOSITION 6.1** *Suppose  $n \in \mathbb{N}$ ,  $r \in \mathbb{Z}_+$ , and  $L$  is an integer with  $L \geq 4n - 2$ . Then the sequence  $\{\phi_j^r\}$ , as defined by (6.13), satisfies*

$$\phi_j^r(x) = \phi(x - j), \quad x \in [2n - 2, 2^r L - 2n + 2], \quad j \in \mathbb{Z}_{2^r L}, \quad (6.16)$$

$$\phi_j^r(x) = \ell_{j-n+1}(x - n + 1), \quad x \in [0, 1], \quad j \in \mathbb{Z}_{2n-1}, \quad (6.17)$$

$$\phi_j^r\left(\frac{k}{2}\right) = \begin{cases} \ell_{j-n+1}\left(\frac{k}{2} - n + 1\right), & k \in \mathbb{Z}_{2n-1}, \\ \phi\left(\frac{k}{2} - j\right), & k = 2n - 2, \dots, \end{cases} \quad j \in \mathbb{Z}_{2n-1}. \quad (6.18)$$

**Proof.** For  $j \in \mathbb{Z}_{2^r L - 2n}$ , (6.16) follows from the definition (6.13) and the finite support property (2.31) of  $\phi$ . It therefore suffices to prove (6.16) for  $j = 2^r L - 2n + 1, \dots, 2^r L$ . But then, from (6.13c) and (6.13a),

$$\phi_j^r(x) = \phi_{2^r L - j}^r(2^r L - x) = \phi(-x + j) = \phi(x - j),$$

by virtue of the symmetry property (2.64) of  $\phi$ .

To prove (6.17), suppose  $x \in [0, 1]$  and  $j \in \mathbb{Z}_{2n-1}$ . Then (6.13a), (2.9) and (2.31) yield

$$\begin{aligned} \phi_j^r(x) &= \sum_{k \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(k - n + 1) \phi(x - k) \\ &= \sum_k \ell_{j-n+1}(k - n + 1) \phi(x - k) = \ell_{j-n+1}(x - n + 1), \end{aligned}$$

by virtue the fact that  $\phi$  satisfies the polynomial reproduction property (2.63).

For the proof of (6.18), we first let  $k \in \mathbb{Z}_{2n-1}$  and  $j \in \mathbb{Z}_{2n-1}$ . Then, using consecutively (6.13a), (2.9), (6.15), (2.31) and (2.63), we get

$$\phi_j^r\left(\frac{k}{2}\right) = \phi\left(\frac{k}{2} - j\right) + \sum_{i=-2n+2}^{-1} \ell_{j-n+1}(i - n + 1) \phi\left(\frac{k}{2} - i\right) \quad (6.19)$$

$$\begin{aligned} &= \sum_{i \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(i - n + 1) \phi\left(\frac{k}{2} - i\right) \\ &= \sum_{i=-2n+2}^{3n-2} \ell_{j-n+1}(i - n + 1) \phi\left(\frac{k}{2} - i\right) \\ &= \sum_i \ell_{j-n+1}(i - n + 1) \phi\left(\frac{k}{2} - i\right) \quad (6.20) \end{aligned}$$

$$= \ell_{j-n+1}\left(\frac{k}{2} - n + 1\right). \quad (6.21)$$

Next, for  $k \geq 2n$  and  $j \in \mathbb{Z}_{2n-1}$ , we see that (6.19) holds again, and the bottom part of (6.18) therefore follows, since  $\phi\left(\frac{k}{2} - i\right) = 0$ ,  $i = -2n + 2, \dots, -1$ , by virtue of the interpolatory property (2.32) and (6.14).

To complete the proof of (6.18), it remains to show that for  $j \in \mathbb{Z}_{2n-1}$ , we have

$$\phi\left(\frac{k}{2} - j\right) = \ell_{j-n+1}\left(\frac{k}{2} - n + 1\right), \quad k \in \{2n - 2, 2n - 1\}. \quad (6.22)$$

For  $k = 2n - 2$ , the property (6.22) is a consequence of (2.32) and (2.9), whereas for  $k = 2n - 1$  we use (2.56) and the middle line of (2.12) to deduce that

$$\phi\left(\frac{k}{2} - j\right) = d_{2n-1-2j} = \ell_{j-n+1}\left(\frac{1}{2}\right) = \ell_{j-n+1}\left(\frac{k}{2} - n + 1\right). \quad \blacksquare$$

We proceed to show in the following section that the sequence  $\{\phi_j^{(r)}\}$  defined by (6.13) is refinable.

## 6.2 The refinability of the sequence $\{\phi_j^r\}$

Analogous to the bi-infinite case, our proof in Section 6.3 below of the convergence of the subdivision scheme will depend on the refinability of the sequence  $\{\phi_j^r\}$ , as proved in this section.

**THEOREM 6.2** *The sequence  $\{\phi_j^r\}$ , as defined in (6.13), satisfies the properties (6.2) – (6.6).*

**Proof.** The properties (6.2) – (6.4) are immediate consequences of the properties (2.32) and (2.31) of  $\phi$ .

To prove (6.5), we choose  $p \in \pi_{2n-1}$ , and assume first that  $x \in [0, 2n - 2]$ . Using (6.3), (6.13a), (6.13b), the polynomial reproduction property (2.63), finite support (2.31) of  $\phi$ , and (2.10) consecutively, we get

$$\begin{aligned} \sum_{j \in \mathbb{Z}_{2^r L}} p(j) \phi_j^r(x) &= \sum_{j \in \mathbb{Z}_{4n-4}} p(j) \phi_j^r(x) \\ &= \sum_{j \in \mathbb{Z}_{4n-4}} p(j) \phi(x - j) + \sum_{k=-2n+2}^{-1} \left[ \sum_{j \in \mathbb{Z}_{2n-1}} p(j) \ell_{j-n+1}(k - n + 1) \right] \phi(x - k) \\ &= \sum_{j=-2n+2}^{4n-4} p(j) \phi(x - j) = \sum_j p(j) \phi(x - j) = p(x). \end{aligned}$$

For  $x \in [2n - 2, 2^r L - 2n + 2]$ , the property (6.5) follows from (6.16) and (6.12). Similar arguments establish polynomial reproduction for  $x \in (2^r L - 2n + 2, 2^r L]$ .



To prove the refinability (6.6), assume first that  $j \in \mathbb{Z}_{2n-1}$  and  $x \in [0, 2n-2)$ . Then, (6.13a) and (2.9), imply

$$\phi_j^r(x) = \sum_{k \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(k-n+1) \phi(x-k).$$

Now, use the refinement equation (2.61), together with (2.31), to obtain

$$\phi_j^r(x) = \sum_{i=-2n+2}^{6n-6} \left[ \sum_{k \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(k-n+1) \phi\left(\frac{i}{2}-k\right) \right] \phi(2x-i). \quad (6.23)$$

For the first part of the sum in (6.23) we get, from (2.31), (6.14), (6.15), the polynomial reproduction (2.63), and definition (6.13a),

$$\begin{aligned} & \sum_{i \in \mathbb{J}_{2n-1}} \left[ \sum_{k \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(k-n+1) \phi\left(\frac{i}{2}-k\right) \right] \phi(2x-i) \\ &= \sum_{i \in \mathbb{J}_{2n-1}} \left[ \sum_k \ell_{j-n+1}(k-n+1) \phi\left(\frac{i}{2}-k\right) \right] \phi(2x-i) \\ &= \sum_{i \in \mathbb{J}_{2n-1}} \ell_{j-n+1}\left(\frac{i}{2}-n+1\right) \phi(2x-i) \\ &= \sum_{i=-2n+2}^{-1} \ell_{j-n+1}\left(\frac{i}{2}-n+1\right) \phi(2x-i) + \sum_{i \in \mathbb{Z}_{2n-1}} \ell_{j-n+1}\left(\frac{i}{2}-n+1\right) \phi_i^{r+1}(2x) \\ & \quad - \sum_{k=-2n+2}^{-1} \left[ \sum_{i \in \mathbb{Z}_{2n-1}} \ell_{j-n+1}\left(\frac{i}{2}-n+1\right) \ell_{i-n+1}(k-n+1) \right] \phi(2x-k) \\ &= \sum_{i \in \mathbb{Z}_{2n-1}} \ell_{j-n+1}\left(\frac{i}{2}-n+1\right) \phi_i^{r+1}(2x) = \sum_{i \in \mathbb{Z}_{2n-1}} \phi_j^r\left(\frac{i}{2}\right) \phi_i^{r+1}(2x), \end{aligned} \quad (6.24)$$

having also used the polynomial reproduction (2.10), and (6.18).

For the remaining part of the sum in (6.23), we use the interpolatory properties (2.9) and (2.32), as well as (6.14) and (6.18), to obtain

$$\begin{aligned} & \sum_{i=2n}^{6n-6} \left[ \sum_{k \in \mathbb{J}_{2n-1}} \ell_{j-n+1}(k-n+1) \phi\left(\frac{i}{2}-k\right) \right] \phi(2x-i) \\ &= \sum_{i=2n}^{6n-6} \phi\left(\frac{i}{2}-j\right) \phi(2x-i) \\ &= \sum_{i=2n}^{6n-6} \phi_j^r\left(\frac{i}{2}\right) \phi_i^{r+1}(2x) = \sum_{i=2n}^{2^{r+1}L} \phi_j^r\left(\frac{i}{2}\right) \phi_i^{r+1}(2x), \end{aligned} \quad (6.25)$$

where we also used the definition (6.13b), and the inequality  $L \geq 4n - 2$ . Combining (6.23), (6.24) and (6.25) then yields the result (6.6) for  $j \in \mathbb{Z}_{2n-1}$  and  $x \in [0, 2n - 2]$ .

Next, for  $j \in \mathbb{Z}_{2n-1}$  and  $x \in [2n - 2, 2n - 1 + j]$ , we use Theorems 2.3, as well as (6.14), (2.61), (6.18) and (6.14), to get

$$\begin{aligned} \phi_j^r(x) = \phi(x - j) &= \sum_k \phi\left(\frac{k}{2} - j\right) \phi(2x - k) \\ &= \sum_{k=2n-2}^{4n-2+2j} \phi\left(\frac{k}{2} - j\right) \phi(2x - k) \\ &= \sum_{k=2n-2}^{4n-2+2j} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) \\ &= \sum_{k \in \mathbb{Z}_{2r+1L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x), \end{aligned}$$

since  $L \geq 4n - 2$ , thereby establishing (6.6) for this subcase.

For  $j \in \mathbb{Z}_{2n-1}$  and  $x \in [2n - 1 + j, 2^r L]$ , we deduce from (6.3), (6.18), (2.32) and (6.14) that

$$\begin{aligned} \sum_{k \in \mathbb{Z}_{2r+1L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) &= \sum_{k=2n+2j}^{4n-3+2j} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) \\ &= \sum_{k=2n+2j}^{4n-3+2j} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) = 0 = \phi_j^r(x), \end{aligned}$$

by virtue of the top part of (6.3). Hence, we have established (6.6) for all  $j \in \mathbb{Z}_{2n-1}$ .

Now consider the case  $j \in \{2n, \dots, 2^r L - 2n\}$  and  $x \in (-2n + 1 + j, 2n - 1 + j)$ . We use definition (6.13b), (2.61), (2.31), as well as (6.14) and the finite support property (6.3), to find that

$$\begin{aligned} \phi_j^r(x) = \phi(x - j) &= \sum_{k \in \{2j + \mathbb{J}_{2n-1}\}} \phi\left(\frac{k}{2} - j\right) \phi(2x - k) \\ &= \sum_{k \in \{2j + \mathbb{J}_{2n-1}\}} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) \\ &= \sum_{k \in \mathbb{Z}_{2r+1L}} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) \\ &= \sum_{k \in \mathbb{Z}_{2r+1L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x), \end{aligned}$$

based on the inequalities  $-2n + 1 + 2j \geq 2n + 1$ , and  $2n - 1 + 2j \leq 2^{r+1}L - 2n - 1$ .

For  $j \in \{2n, \dots, 2^rL - 2n\}$  and  $x \in [0, -2n + 1 + j]$ , we use (6.3), (6.13b), (2.32) and (6.15) to get

$$\sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) = \sum_{k \in \mathbb{Z}_{2j-2n}} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) = 0 = \phi_j^r(x).$$

Similarly, for  $j \in \{2n, \dots, 2^rL - 2n\}$  and  $x \in [2n - 1 + j, 2^rL]$ , with (6.14), we have

$$\sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) = \sum_{k=2n+2j}^{2^{r+1}L} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) = 0 = \phi_j^r(x).$$

Hence (6.6) also holds for  $j \in \{2n, \dots, 2^rL - 2n\}$ .

Finally, let  $j \in \{2^rL - 2n + 1, \dots, 2^rL\}$  and  $x \in [0, 2^rL]$ . Then (6.13c), together with the fact that (6.6) holds for  $j \in \mathbb{Z}_{2n-1}$ , gives

$$\begin{aligned} \phi_j^r(x) &= \phi_{2^rL-j}^r(2^rL - x) \\ &= \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_{2^rL-j}^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2^{r+1}L - 2x) \\ &= \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_{2^rL-j}^r\left(2^rL - \frac{k}{2}\right) \phi_{2^{r+1}L-k}^{r+1}(2^{r+1}L - 2x) \\ &= \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_{2^{r+1}L-k}^{r+1}(2^{r+1}L - 2x). \end{aligned}$$

We claim that

$$\phi_{2^{r+1}L-k}^{r+1}(2^{r+1}L - 2x) = \phi_k^{r+1}(2x), \quad x \in [0, 2^rL], \quad k \in \mathbb{Z}_{2^{r+1}L}, \quad (6.26)$$

which, if true, completes the proof of the theorem. Definition (6.13c) implies that (6.26) is true for  $k \in \mathbb{Z}_{2n-1} \cup \{2^rL - 2n + 1, \dots, 2^rL\}$ , whereas, if  $k \in \{2n, \dots, 2^rL - 2n\}$ , we find that (6.13b) and (2.55) yield, for  $x \in [0, 2^rL]$ ,

$$\phi_{2^{r+1}L-k}^{r+1}(2^{r+1}L - 2x) = \phi(k - 2x) = \phi(2x - k) = \phi_k^{r+1}(2x). \quad \blacksquare$$

**Remark:** Observe from (6.16) and (6.13a), (6.13b), together with the support properties (6.3) and (2.31), that

$$\phi_j^r(x) = \phi(x - j), \quad x \in [2n - 2, 2^rL], \quad j \in \mathbb{Z}_{2^rL-2n}. \quad (6.27)$$

## 6.3 Convergence of the modified subdivision scheme

We now prove the analogue of the results (2.27) and (2.28) Theorem 2.2 for *finite* subdivision sequences.

**THEOREM 6.3** *For each initial sequence  $\mathbf{c} = \{c_j\} \in \mathbf{M}_0$  and with  $\phi_j = \phi_j^0$  defined as in (6.13), the subdivision scheme (6.7), (6.8) converges to the function*

$$\Phi_L(x) = \sum_{j \in \mathbb{Z}_L} c_j \phi_j(x), \quad x \in [0, L], \quad (6.28)$$

*in the sense that*

$$c_k^{(r)} = \Phi_L\left(\frac{k}{2^r}\right), \quad k \in \mathbb{Z}_{2^r L}, \quad r \in \mathbb{Z}_+. \quad (6.29)$$

**Proof.** Repeatedly using (6.6), (6.7), (6.8), and eventually (6.4), for  $k \in \mathbb{Z}_{2^r L}$  and  $r \in \mathbb{Z}_+$ , we obtain

$$\begin{aligned} \Phi_L\left(\frac{k}{2^r}\right) &= \sum_{j \in \mathbb{Z}_L} c_j \phi_j^0\left(\frac{k}{2^r}\right) \\ &= \sum_{j \in \mathbb{Z}_L} c_j \sum_{i \in \mathbb{Z}_{2L}} \phi_j^0\left(\frac{i}{2}\right) \phi_i^1\left(\frac{k}{2^{r-1}}\right) \\ &= \sum_{i \in \mathbb{Z}_{2L}} c_i^{(1)} \phi_i^1\left(\frac{k}{2^{r-1}}\right) \\ &= \dots \\ &= \sum_{i \in \mathbb{Z}_{2^r L}} c_i^{(r)} \phi_i^r(k) = c_k^{(r)}. \quad \blacksquare \end{aligned}$$

Analogous to the subdivision scheme with mask (2.16), we proceed in the following section to derive an explicit formulation of the boundary-adapted subdivision scheme (6.7), (6.8).

## 6.4 An explicit formulation

Let  $\mathbf{c} \in \mathbf{M}_r$ . Then, from the subdivision operator definition (6.7), we obtain

$$(S_r \mathbf{c})_{2j+1} = \sum_{k \in \mathbb{Z}_{2^r L}} \phi_k^r\left(j + \frac{1}{2}\right) c_k, \quad j \in \mathbb{Z}_{2^r L-1}, \quad (6.30)$$

and thus, using also (6.13b), (6.15) and the top of (6.18), we get

$$(S_r c)_{2j+1} = \sum_{k \in \mathbb{Z}_{2n-1}} \ell_{k-n+1} \left( j + \frac{1}{2} - n + 1 \right) c_k, \quad j \in \mathbb{Z}_{n-1}. \quad (6.31)$$

Next, we claim that

$$(S_r c)_{2j+1} = \sum_{k \in \{j+\mathbb{J}_n\}} \ell_{k-j} \left( \frac{1}{2} \right) c_k, \quad j = n, \dots, 2^r L - 2n. \quad (6.32)$$

Indeed, if  $j \in \{n, \dots, 2n-3\}$ , then (6.30), (6.3), (6.18), (6.13b), (2.56) and the last line of (2.12) yield

$$(S_r c)_{2j+1} = \sum_{k \in \mathbb{Z}_{4n-4}} \phi \left( j + \frac{1}{2} - k \right) c_k = \sum_{k \in \{j+\mathbb{J}_n\}} d_{2j+1-2k} c_k,$$

and (6.32) then follows from the middle line of (2.12) for  $j \in \{n, \dots, 2n-3\}$ . Similarly, if  $j \in \{2n-2, \dots, 2^r L - 2n\}$ , we additionally use (6.16) to get

$$(S_r c)_{2j+1} = \sum_{k \in \mathbb{Z}_{2^r L}} \phi \left( j + \frac{1}{2} - k \right) c_k = \sum_{k \in \{-n+\mathbb{J}_n\}} d_{2j+1-2k} c_k,$$

which, together with the middle line of (2.12), then proves (6.32) for  $j \in \{2n-2, \dots, 2^r L - 2n\}$ .

For  $j \in \{2^r L - 2n + 1, \dots, 2^r L - 1\}$ , we first note the symmetry

$$\phi_k^r(x) = \phi_{2^r L - k}^r(2^r L - x), \quad x \in [0, 2^r L], \quad k \in \mathbb{Z}_{2^r L}, \quad (6.33)$$

which follows from (6.13c), and the fact that, for  $k \in \{2n, \dots, 2^r L - 2n\}$ , (6.13b) and (2.64) give

$$\phi_{2^r L - k}^r(2^r L - x) = \phi(k - x) = \phi(x - k) = \phi_k^r(x).$$

Thus, from (6.33),

$$\begin{aligned} \phi_k^r \left( j + \frac{1}{2} \right) &= \phi_{2^r L - k}^r \left( (2^r L - 1 - j) + \frac{1}{2} \right), & k \in \mathbb{Z}_{2^r L}, \\ &= \phi_{2^r L - k}^r(2^r L - 1 - j), & j = 2^r L - 2n + 1, \dots, 2^r L - 1. \end{aligned} \quad (6.34)$$

Combining (6.31), (6.32), (6.34), and using (6.10), we find that the subdivision scheme (6.7), (6.8) has, for a given initial sequence  $\mathbf{c}^{(0)} = \mathbf{c} \in \mathbf{M}_0$ , the explicit formulation

$$\left. \begin{aligned} \mathbf{c}_{2j}^{(r+1)} &= \mathbf{c}_j^{(r)}, & j \in \mathbb{Z}_{2^r L}, \\ \mathbf{c}_{2j+1}^{(r+1)} &= \sum_{k \in \mathbb{Z}_{2^r L}} \mathbf{a}_{j,k}^{(r)} \mathbf{c}_k^{(r)}, & j \in \mathbb{Z}_{2^r L-1}, \end{aligned} \right\} \quad r \in \mathbb{Z}_+, \quad (6.35)$$

where

$$\mathbf{a}_{j,k}^{(r)} = \begin{cases} \ell_{k-n+1}(j + \frac{3}{2} - n), & k \in \mathbb{Z}_{2n-1}, \\ 0, & k = 2n, \dots, 2^r L, \end{cases} \quad j \in \mathbb{Z}_{n-2}, \quad (\text{if } n \geq 2), \quad (6.36)$$

$$\mathbf{a}_{j,k}^{(r)} = \begin{cases} \ell_{k-j}(\frac{1}{2}), & k \in \{j + \mathbb{J}_n\}, \\ 0, & k \in \mathbb{Z}_{2^r L} \setminus \{j + \mathbb{J}_n\}, \end{cases} \quad j = n-1, \dots, 2^r L - n, \quad (6.37)$$

$$\mathbf{a}_{j,k}^{(r)} = \mathbf{a}_{2^r L-1-j, 2^r L-k}^{(r)}, \quad k \in \mathbb{Z}_{2^r L}, \quad j = 2^r L - n + 1, \dots, 2^r L - 1. \quad (6.38)$$

Explicit formulations of (6.36) and (6.37) are now obtained by using a calculation similar to the one which yielded the middle line of (2.16). For  $k \in \mathbb{Z}_{2n-1}$ , and  $j \in \mathbb{Z}_{n-2}$ , we have

$$\ell_{k-n+1}(j + \frac{3}{2} - n) = \frac{(-1)^{j+k}}{2^{4n-3}} \frac{1}{2j+1-2k} \frac{(2j+1)!(4n-3-2j)!}{(2n-2-j)!(2n-1-k)!j!k!}, \quad (6.39)$$

whereas for  $k \in \{j + \mathbb{J}_n\}$ , and  $j = n-1, \dots, 2^r L - n$ ,

$$\ell_{k-j}(\frac{1}{2}) = \frac{n}{2^{4n-3}} \binom{2n-1}{n-1} \frac{(-1)^{j+k}}{2j+1-2k} \binom{2n-1}{n+j-k}. \quad (6.40)$$

For example, if  $n = 2$ , (6.39) and (6.40) in (6.36) give

$$\mathbf{a}_{0,k}^{(r)} = \begin{cases} \frac{5}{16}, & k = 0, \\ \frac{15}{16}, & k = 1, \\ -\frac{5}{16}, & k = 2, \\ \frac{1}{16}, & k = 3, \\ 0, & k = 4, \dots, 2^r L. \end{cases} \quad (6.41)$$

In Figure 6.2, we have applied the modified subdivision scheme (6.35), with mask based on (6.41) and (2.18) to a specific choice of finite (non-periodic) initial sequence  $\mathbf{c}$ . Note how there are no edge effects at the boundaries of the initial sequence.

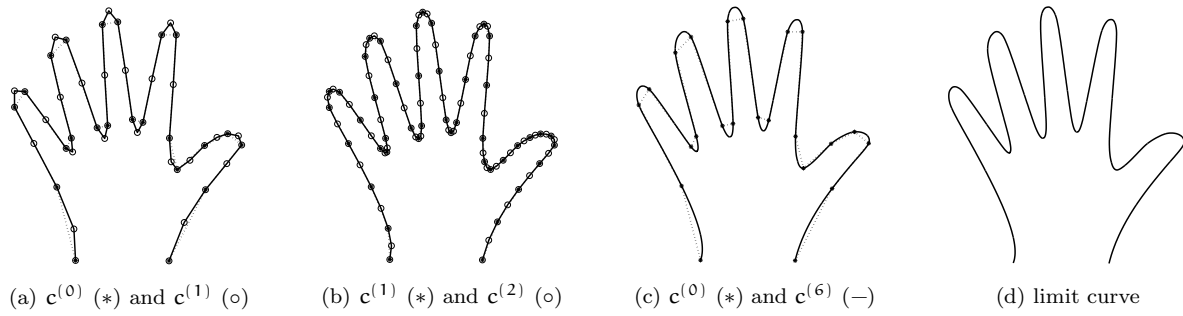


Figure 6.2: The adapted Dubuc–Deslauriers subdivision scheme (6.35) with mask (6.41)

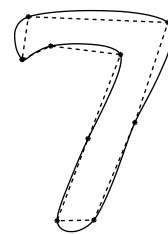




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# Interpolation wavelets on an interval

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Following Section 4 of [21], we show here how the refinable sequence  $\{\phi_j^r\}$  of Section 6.2 can be used to explicitly construct interpolation wavelets on an interval. Our definition in (7.10) below coincides, in the inner region bounded away from the endpoints, with the definition of interpolation wavelets on  $\mathbb{R}$  as given in e.g. [12, equation (1.11)], [47, p 300] and [63, p 193].

## 7.1 Background

Alternative approaches to the construction of wavelets on an interval include work by Daubechies [17, Section 10.7] and Cohen, Daubechies and Vial [14], in which periodization and related methods are used for the construction of orthonormal wavelet bases on an interval. In the spline setting, explicit constructions of symmetric biorthogonal spline wavelets on an interval, as well as the corresponding decomposition and reconstruction algorithms, appear in Chui and Quak [13], Quak and Weyrich [59], Chui and de Villiers [11] and Chui [9, Section 7.3.2].

The connection between the compactly supported orthonormal wavelets of Daubechies [16, 17] and Dubuc–Deslauriers subdivision has been noted by several authors (see e.g. [51, Section 3]), and exploited for the construction of biorthogonal interpolatory wavelets by Beylkin and Saito [4], and Bertoluzza and Naldi [2, 3]. A further study of the relationship between interpolation processes and wavelets construction appears in the paper by Lee, Sharma and Tan [46].

The idea, as used in this chapter, to construct interpolation wavelets by means of a non-orthogonal linear space decomposition and an interpolation operator, has been studied for wavelets on  $\mathbb{R}$  by Chui and Li [12], and used by Sweldens, to construct lifting schemes in [63].

Our interpolation wavelet decomposition and reconstruction algorithms for finite data sets, as given by (7.32), (7.33) below and (6.36) – (6.40) in Chapter 6, are identical to those derived by Aràndiga, Donat and Harten [1, equations (55) and (56)]. Making use of ideas developed by Harten [39, 40], these authors derive their equations from a general framework—our approach has the advantage that it allows an explicit construction of the underlying refinable sequence  $\{\phi_j^r\}$  as demonstrated by our equation (6.13) in the previous chapter. This is perhaps closer to the unpublished work of Donoho [29], in the sense that both are based on polynomial extrapolation. An essential difference however, lies in the way in which the associated nested sequence of linear spaces  $\{V_r\}$  is defined. While Donoho’s construction is consistently based on a polynomial extrapolation operator, we define the linear space  $V_r$  as the span of the sequence  $\{\phi_j^r\}$ . The nesting property of the  $\{V_r\}$  then follows from the refinability of  $\{\phi_j^r\}$ , as proved in Section 7.2 below.

Note that the wavelets constructed in this manner do not have vanishing moments—the mean of our wavelet is not zero. In this regard our wavelets are similar to those mentioned by Mallat [47, p 301]. We show however that the corresponding interpolation wavelet space can be characterized in terms of a projection operator which is exact on polynomials, therefore the interpolation wavelet coefficients of a function are relatively small in those regions where the function exhibits local polynomial-like behavior. For the construction of (non-orthogonal) interpolation wavelets on the interval with vanishing moments, we refer to Donoho [30].

Finally, Schröder and Sweldens [62] develop algorithms implementing interpolation wavelets on an interval without providing detailed proofs.

## 7.2 Decomposition based on interpolation

The main result of this chapter is the direct sum (non-orthogonal) space decomposition of Theorem 7.1 below, by virtue of which we obtain compactly supported symmetric interpolation wavelets.

Let  $n \in \mathbb{N}$  and  $L \geq 4n - 2$  be as in Chapter 6, and let  $R$  be a given positive integer. We define the linear space sequence  $\{V_r : r \in \mathbb{Z}_R\}$  by

$$V_r = \text{span}\{\phi_j^r(2^{r-R}\cdot) : j \in \mathbb{Z}_{2^r L}\}, \quad r \in \mathbb{Z}_R, \quad (7.1)$$

and the linear operator sequence  $\{P_r : r \in \mathbb{Z}_R\}$  for  $P_r : C[0, 2^R L] \rightarrow V_r$ , by

$$(P_r f)(x) = \sum_{j \in \mathbb{Z}_{2^r L}} f(2^{R-r}j) \phi_j^r(2^{r-R}x), \quad x \in [0, 2^R L], \quad r \in \mathbb{Z}_R, \quad (7.2)$$

with  $\phi = \phi_n^D$  denoting, as in Chapter 6, the Dubuc–Deslauriers interpolatory refinable function of Theorem 2.3. It follows from (6.4) that  $P_r$  is an interpolation operator, which means that, for each  $f \in C[0, 2^R L]$ ,

$$(P_r f)(2^{R-r}j) = f(2^{R-r}j), \quad j \in \mathbb{Z}_{2^r L}, \quad r \in \mathbb{Z}_R. \quad (7.3)$$

Also, a proof based on (6.4) shows that  $P_r$  is a projection on  $V_r$ . Thus,

$$P_r f = f, \quad f \in V_r, \quad r \in \mathbb{Z}_R. \quad (7.4)$$

Furthermore, (6.5) gives

$$\sum_{j \in \mathbb{Z}_{2^r L}} p(2^{R-r}j) \phi_j^r(2^{r-R}x) = p(x), \quad x \in [0, 2^r L], \quad p \in \pi_{2n-1}, \quad r \in \mathbb{Z}_R, \quad (7.5)$$

by virtue of which

$$\pi_{2n-1} \subset V_r, \quad r \in \mathbb{Z}_R, \quad (7.6)$$

where here  $\pi_{2n-1}$  is restricted to the interval  $[0, 2^r L]$ .

Since (6.6) yields the refinement equation,

$$\phi_j^r(2^{r-R}x) = \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^r L], \quad j \in \mathbb{Z}_{2^r L}, \quad r \in \mathbb{Z}_{R-1}, \quad (7.7)$$

we have the nesting property

$$V_r \subset V_{r+1}, \quad r \in \mathbb{Z}_{R-1}. \quad (7.8)$$

Based on (6.4) one can prove that, for each fixed  $r$ , the set  $\{\phi_j^r(2^{r-R} \cdot) : j \in \mathbb{Z}_{2^r L}\}$  is linearly independent on  $[0, 2^r L]$ , and thus, from (7.1), we have

$$\dim V_r = 2^r L + 1, \quad r \in \mathbb{Z}_R. \quad (7.9)$$

Now define the sequence  $\{\psi_j^r\} = \{\psi_j^r : j \in \mathbb{Z}_{2^r L-1}, r \in \mathbb{Z}_{R-1}\}$  by

$$\psi_j^r(x) = \phi_{2j+1}^{r+1}(x), \quad x \in [0, 2^r L], \quad j \in \mathbb{Z}_{2^r L-1}, \quad r \in \mathbb{Z}_{R-1}, \quad (7.10)$$

with corresponding linear spaces

$$W_r = \text{span}\{\psi_j^r(2^{r+1-R} \cdot) : j \in \mathbb{Z}_{2^r L-1}\}, \quad r \in \mathbb{Z}_{R-1}. \quad (7.11)$$

From (7.10) and (7.1) follows that

$$W_r \subset V_{r+1}, \quad r \in \mathbb{Z}_{R-1}. \quad (7.12)$$

If  $U$ ,  $V$  and  $W$  are linear spaces, we use the direct sum notation  $U = V \oplus W$  to denote that fact that, for each  $f \in U$ , there exist  $g \in V$  and  $h \in W$  such that  $f = g + h$ , and with  $g$  and  $h$  uniquely determined by  $f$ .

The following direct sum decomposition result holds.

**THEOREM 7.1**

$$V_{r+1} = V_r \oplus W_r, \quad r \in \mathbb{Z}_{R-1}, \quad (7.13)$$

with  $V_r$  and  $W_r$  defined by (7.1) and (7.11).

To prove Theorem 7.1, we first introduce the linear spaces  $U_r$  and  $X_r$ , where

$$U_r = \{f - P_r f : f \in V_{r+1}\}, \quad r \in \mathbb{Z}_{R-1}, \quad (7.14)$$

with  $P_r$  defined by (7.2), and

$$X_r = \{f \in V_{r+1} : f(2^{R-r}j) = 0, \quad j \in \mathbb{Z}_{2^r L}\}, \quad r \in \mathbb{Z}_{R-1}, \quad (7.15)$$

in terms of which the following preliminary result holds.

**PROPOSITION 7.2** *The linear spaces  $V_r, W_r, U_r$  and  $X_r$ , as defined by (7.1), (7.11), (7.14) and (7.15), satisfy*

$$(a) \quad W_r = U_r = X_r, \quad r \in \mathbb{Z}_{R-1}; \quad (7.16)$$

$$(b) \quad V_r \cap W_r = \{0\}, \quad r \in \mathbb{Z}_{R-1}. \quad (7.17)$$

**Proof.** (a) First, we show that  $U_r = X_r$ . If  $g \in U_r$ , then (7.2) and (7.3) yield  $g(2^{R-r}j) = 0$ ,  $j \in \mathbb{Z}_{2^r L}$ , i.e.  $g \in X_r$ , so that  $U_r \subset X_r$ . If  $g \in X_r$ , then (7.2) and (7.15) imply  $P_r g = 0$ ; hence,  $g = f - P_r f$  with  $f = g$ , and thus, since also  $f \in X_r \subset V_{r+1}$  from (7.15), we have from (7.14) that  $g \in U_r$ . Consequently,  $X_r \subset U_r$ .

Next, we prove that  $W_r \subset X_r$ . If  $g \in W_r$ , there exists a coefficient sequence  $\{c_0, \dots, c_{2^r L-1}\}$  such that  $g(x) = \sum_{k \in \mathbb{Z}_{2^r L-1}} c_k \psi_k^r(2^{r+1-R}x)$ ,  $x \in [0, 2^r L]$ . But then, from (7.10), we have

$$g(2^{R-r}j) = \sum_{k \in \mathbb{Z}_{2^r L-1}} c_k \phi_{2k+1}^{r+1}(2j) = 0, \quad j \in \mathbb{Z}_{2^r L},$$

by virtue of (6.4). Definition (7.15) then implies that  $g \in X_r$ . So,  $W_r \subset X_r$ .

We now show that  $U_r \subset W_r$ , thereby completing the proof of (7.16). Suppose therefore  $g \in U_r$ , so that, from (7.14) and (7.1), there exists a coefficient sequence  $\{c_j : j \in \mathbb{Z}_{2^{r+1}L}\}$  such that the function  $f \in V_{r+1}$  given by

$$f(x) = \sum_{j \in \mathbb{Z}_{2^{r+1}L}} c_j \phi_j^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^R L], \quad (7.18)$$

satisfies the equation

$$g(x) = f(x) - (P_r f)(x), \quad x \in [0, 2^R L]. \quad (7.19)$$

But, from (7.2), (7.18), (6.4) and (6.6), we obtain

$$\begin{aligned}
 (P_r f)(x) &= \sum_{j \in \mathbb{Z}_{2^{r+1}L}} \left[ \sum_{k \in \mathbb{Z}_{2^r L}} \phi_j^{r+1}(2k) \phi_k^r(2^{r-R}x) \right] c_j \\
 &= \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j} \phi_j^r(2^{r-R}x) \\
 &= \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j} \sum_{k \in \mathbb{Z}_{2^{r+1}L}} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2^{r+1-R}x) \\
 &= \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j} \phi_{2j}^{r+1}(2^{r+1-R}x) \\
 &\quad + \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j} \sum_{k \in \mathbb{Z}_{2^{r+1}L-1}} \phi_j^r\left(k + \frac{1}{2}\right) \phi_{2k+1}^{r+1}(2^{r+1-R}x). \tag{7.20}
 \end{aligned}$$

Also, from (7.18),

$$f(x) = \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j} \phi_{2j}^{r+1}(2^{r+1-R}x) + \sum_{j \in \mathbb{Z}_{2^{r+1}L-1}} c_{2j+1} \phi_{2j+1}^{r+1}(2^{r+1-R}x). \tag{7.21}$$

Combining (7.19), (7.20) and (7.21) yields

$$g(x) = \sum_{j \in \mathbb{Z}_{2^{r+1}L-1}} \left[ c_{2j+1} - \sum_{k \in \mathbb{Z}_{2^r L}} c_{2k} \phi_k^r\left(j + \frac{1}{2}\right) \right] \phi_{2j+1}^{r+1}(2^{r+1-R}x), \tag{7.22}$$

which, together with (7.11), (7.10), implies that  $g \in W_r$ . Thus  $U_r \subset W_r$ .

(b) Suppose  $f \in V_r \cap W_r$ . Then (7.4) and (7.2) imply that, for  $x \in [0, 2^r L]$ ,

$$f(x) = (P_r f)(x) = \sum_{j \in \mathbb{Z}_{2^r L}} f(2^{R-r}j) \phi_j^r(2^{r-R}x) = 0,$$

after noting from (7.16) that  $f \in X_r$ , and then using the definition (7.15). ■

We can now prove Theorem 7.1.

**Proof.** Let  $r \in \mathbb{Z}_{R-1}$  be fixed. Take any  $f \in V_{r+1}$  and define  $g = P_r f$  and  $h = f - P_r f$ . Then (7.2) implies that  $g \in V_r$ , whereas (7.14) and (7.16) imply that  $h \in W_r$ . We have therefore shown that there exist functions  $g \in V_r$  and  $h \in W_r$  such that  $f = g + h$ . It remains to be proved that  $g$  and  $h$  are uniquely determined by  $f$ . But, if  $g_0 \in V_r$  and  $h_0 \in W_r$  are such that  $f = g_0 + h_0$ , then  $u = g_0 - g \in V_r$  and  $v = h - h_0 \in W_r$ , with

$u = v$ . Thus  $u \in V_r \cap W_r$  and  $v \in V_r \cap W_r$ , and the desired uniqueness result follows from (7.17).  $\blacksquare$

From (7.13) one concludes that  $\dim W_r = \dim V_{r+1} - \dim V_r$ , so that (7.9) leads to

$$\dim W_r = 2^r L, \quad r \in \mathbb{Z}_{R-1}. \quad (7.23)$$

Hence, based on the definition (7.11), we conclude that the set  $\{\psi_j^r(2^{r+1-R} \cdot) : j \in \mathbb{Z}_{2^r L-1}\}$  is linearly independent on  $[0, 2^R L]$ , and therefore is a basis for  $W_r$ .

We have therefore established, for each fixed  $r \in \mathbb{Z}_{R-1}$ , an *interpolation wavelet basis*  $\{\psi_j^r(2^{r+1-R} \cdot) : j \in \mathbb{Z}_{2^r L-1}\}$  for the *interpolation wavelet space*  $W_r$ . The elements of the sequence  $\{\psi_j^r\}$  are called *interpolation wavelets*. Examples for  $n = 2$  are plotted in Figure 7.1.

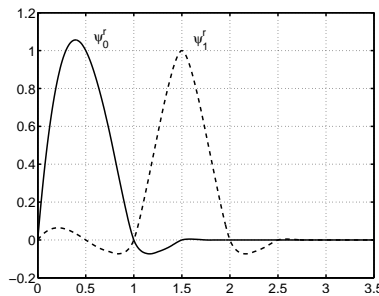


Figure 7.1: Boundary interpolation wavelets with  $n = 2$ .

In the next section, we derive the corresponding wavelet decomposition and reconstruction algorithms.

### 7.3 Decomposition and reconstruction algorithms

The symmetric interpolation wavelets of the previous section lead to finite decomposition and reconstruction formulas with rational coefficients given by the values of certain Lagrange polynomials at half integers, as described in the section below. We then comment on a connection between interpolation wavelet decomposition on an interval and

the adapted subdivision of Chapter 6 and also on a property analogous to the vanishing moments property usually associated with wavelets.

To obtain the decomposition algorithm, let  $r \in \mathbb{Z}_{R-1}$  be fixed, and suppose  $f_{r+1} \in V_{r+1}$  is given by

$$f_{r+1}(x) = \sum_{j=0}^{2^{r+1}L} c_j^{(r+1)} \phi_j^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^R L]. \quad (7.24)$$

According to Theorem 7.1, and for the bases  $\{\phi_j^r\}$  and  $\{\psi_j^r\}$  of  $V_r$  and  $W_r$ , we know that there exist unique coefficient sequences  $\{c_j^{(r)} : j \in \mathbb{Z}_{2^r L}\}$  and  $\{d_j^{(r)} : j \in \mathbb{Z}_{2^r L-1}\}$  such that the functions  $f_r \in V_r$  and  $g_r \in W_r$  defined by

$$f_r(x) = \sum_{j \in \mathbb{Z}_{2^r L}} c_j^{(r)} \phi_j^r(2^{r-R}x), \quad x \in [0, 2^R L], \quad (7.25)$$

and

$$g_r(x) = \sum_{j \in \mathbb{Z}_{2^r L-1}} d_j^{(r)} \psi_j^r(2^{r+1-R}x), \quad x \in [0, 2^R L], \quad (7.26)$$

satisfy

$$f_{r+1} = f_r + g_r. \quad (7.27)$$

In particular, observe that we then have the *interpolation wavelet decomposition*

$$f_r = f_0 + \sum_{j \in \mathbb{Z}_{R-1}} g_j.$$

Moreover,

$$f_r = P_r f_{r+1} \quad (7.28)$$

and

$$g_r = f_{r+1} - P_r f_{r+1}. \quad (7.29)$$

The coefficients  $\{d_j^{(r)}\}$  in (7.26) are called the *interpolation wavelet coefficients*.

Using (7.24), (7.10) and (7.29), and the argument which led to (7.22), we obtain, for all  $x \in [0, 2^R L]$ ,

$$\sum_{j \in \mathbb{Z}_{2^r L-1}} d_j^{(r)} \psi_j^r(2^{r+1-R}x) = \sum_{j \in \mathbb{Z}_{2^r L-1}} \left[ c_{2j+1}^{(r+1)} - \sum_{k \in \mathbb{Z}_{2^r L}} c_{2k}^{(r+1)} \phi_k^r \left( j + \frac{1}{2} \right) \right] \psi_j^r(2^{r+1-R}x).$$



Since  $\{\psi_j^r(2^{r+1-R}\cdot) : j \in \mathbb{Z}_{2^r L-1}\}$  is a linearly independent set on  $[0, 2^R L]$ , we have

$$d_j^{(r)} = c_{2j+1}^{(r+1)} - \sum_{k \in \mathbb{Z}_{2^r L}} c_{2k}^{(r+1)} \phi_k^r(j + \frac{1}{2}), \quad j \in \mathbb{Z}_{2^r L-1}. \quad (7.30)$$

Next, using (7.25), (7.28), (7.2) and (6.4) we get, for all  $x \in [0, 2^R L]$ ,

$$\begin{aligned} \sum_{j \in \mathbb{Z}_{2^r L}} c_j^{(r)} \phi_j^r(2^{r-R}x) &= f_r(x) = (P_r f_{r+1})(x) \\ &= \sum_{j \in \mathbb{Z}_{2^r L}} \left[ \sum_{k \in \mathbb{Z}_{2^{r+1} L}} c_k^{(r+1)} \phi_k^{r+1}(2j) \right] \phi_j^r(2^{r-R}x) \\ &= \sum_{j \in \mathbb{Z}_{2^r L}} c_{2j}^{(r+1)} \phi_j^r(2^{r-R}x). \end{aligned}$$

Since the set  $\{\phi_j^r(2^{r-R}\cdot) : j \in \mathbb{Z}_{2^r L}\}$  is linearly independent on  $[0, 2^R L]$ , we deduce that

$$c_j^{(r)} = c_{2j}^{(r+1)}, \quad j \in \mathbb{Z}_{2^r L}. \quad (7.31)$$

Now observe from (6.30), (6.8) and (6.35) that

$$\phi_k^r(j + \frac{1}{2}) = a_{j,k}^{(r)}, \quad k \in \mathbb{Z}_{2^r L}, \quad j \in \mathbb{Z}_{2^r L-1},$$

which, together with (7.30) and (7.31), then yield, for a given data sequence  $\{c_j^{(R)} : j \in \mathbb{Z}_{2^R L}\}$ , the following interpolation wavelet algorithms:

**Decomposition algorithm:**

$$\left. \begin{aligned} c_j^{(r)} &= c_{2j}^{(r+1)}, & j \in \mathbb{Z}_{2^r L}, \\ d_j^{(r)} &= c_{2j+1}^{(r+1)} - \sum_{k \in \mathbb{Z}_{2^r L}} a_{j,k}^{(r)} c_{2k}^{(r+1)}, & j \in \mathbb{Z}_{2^r L-1}, \end{aligned} \right\} r = R-1, R-2, \dots, 0. \quad (7.32)$$

**Reconstruction algorithm:**

$$\left. \begin{aligned} c_{2j}^{(r+1)} &= c_j^{(r)}, & j \in \mathbb{Z}_{2^r L}, \\ c_{2j+1}^{(r+1)} &= d_j^{(r)} + \sum_{k \in \mathbb{Z}_{2^r L}} a_{j,k}^{(r)} c_k^{(r)}, & j \in \mathbb{Z}_{2^r L-1}, \end{aligned} \right\} r = 0, 1, \dots, R-1. \quad (7.33)$$

Here the coefficient sequence  $\{a_{j,k}^{(r)} : k \in \mathbb{Z}_{2^r L}, j \in \mathbb{Z}_{2^r L-1}, r \in \mathbb{Z}_{R-1}\}$  is defined by (6.36), (6.37), (6.38), with explicit formulations in (6.39) and (6.40).

Suppose  $f \in C[0, 2^R]$ , with the integers  $R$  and  $L$  suitably chosen. The sequence  $\{f_r : r = R - 1, R - 2, \dots, 0\}$  is then defined by (7.28), with  $f_R = f$ , whereas the sequence  $\{g_r : r = R - 1, R - 2, \dots, 0\}$  is defined by (7.29). The coefficient sequences  $\{c_j^{(r)} : j \in \mathbb{Z}_{2^r L}, r = R, R - 1, \dots, 0\}$  and  $\{d_j^{(r)} : j \in \mathbb{Z}_{2^{r-1} L}, r = R - 1, R - 2, \dots, 0\}$  are computed recursively by means of (7.32). In particular, observe that since (7.2) and (7.25) yield

$$\sum_{j \in \mathbb{Z}_{2^R L}} c_j^{(R)} \phi_j^R(x) = f_R(x) = (P_R f)(x) = \sum_{j \in \mathbb{Z}_{2^R L}} f(j) \phi_j^R(x),$$

we have  $c_j^{(R)} = f(j)$ ,  $j \in \mathbb{Z}_{2^R L}$ , and thus, the interpolation wavelet decomposition algorithm (7.32) can, in this context, be rewritten as

$$\left. \begin{aligned} c_j^{(r)} &= f(2^{R-r}j), & j &\in \mathbb{Z}_{2^r L}, \\ d_j^{(r)} &= f(2^{R-r-1}(2j+1)) - \sum_{k \in \mathbb{Z}_{2^{r-1} L}} a_{j,k}^{(r)} f(2^{R-r}k), & j &\in \mathbb{Z}_{2^{r-1} L}. \end{aligned} \right\} \quad r = R - 1, R - 2, \dots, 0. \quad (7.34)$$

The reconstruction is then performed by means of (7.33).

At this stage, it is of interest to point out the following relationship between the interpolation wavelet procedure (7.34), (7.33) and the interpolatory subdivision scheme (6.35).

After the decomposition with (7.34) has been performed, suppose that we set the interpolation wavelet coefficients

$$d_j^{(r)} = 0, \quad j \in \mathbb{Z}_{2^{r-1} L}, \quad r \in \mathbb{Z}_{R-1}, \quad (7.35)$$

at each successive step of the reconstruction phase (7.33). The final reconstructed (and smoothed) function is then

$$f_R(x) = \sum_{j \in \mathbb{Z}_{2^R L}} c_j^{(R)} \phi_j^R(x), \quad x \in [0, 2^R], \quad (7.36)$$

with, as is clear from the top parts of (7.33) and (7.34), and (6.4),

$$f_R(2^R k) = c_{2^R k}^{(R)} = f(2^R k), \quad k \in \mathbb{Z}_L.$$

Now observe that the zero values (7.35) substituted into the bottom part of (7.33) yield precisely the bottom part of the subdivision formula (6.35).

Hence the interpolation wavelet decomposition and reconstruction scheme described above is equivalent to the following interpolatory subdivision scheme:

Choose  $\mathbf{c} = \mathbf{c}^{(0)} = f(2^R \mathbf{k})$ ,  $\mathbf{k} \in \mathbb{Z}_+$ , and apply the interpolatory subdivision scheme (6.35). According to Theorem 3.3, this scheme converges to the limit curve

$$g(x) = \sum_{j \in \mathbb{Z}_L} f(2^R j) \phi_j^0(x), \quad x \in [0, L]. \quad (7.37)$$

Then, using the refinement equation (6.6), as well as (6.7) – (6.8), we get

$$\begin{aligned} g(x) &= \sum_{j \in \mathbb{Z}_L} c_j^{(0)} \phi_j^0(x) \\ &= \sum_{j \in \mathbb{Z}_L} c_j^{(0)} \sum_{k \in \mathbb{Z}_{2L}} \phi_j^0\left(\frac{k}{2}\right) \phi_k^1(2x) \\ &= \sum_{k \in \mathbb{Z}_{2L}} c_k^{(1)} \phi_k^1(2x) = \cdots = \sum_{k \in \mathbb{Z}_{2^R L}} c_k^{(R)} \phi_k^R(2^R x) = f_R(2^R x), \end{aligned}$$

from (7.36). Hence,

$$g(x) = f_R(2^R x), \quad x \in [0, L]. \quad (7.38)$$

The subdivision approach therefore has the significant advantage of providing an efficient iterative procedure for the construction of the function  $f_R$  in (7.36).

Finally, suppose in the decomposition procedure defined by the algorithm (7.34) we have, for some given polynomial  $p \in \pi_{2n-1}$ , and for a fixed  $r \in \{R-1, R-2, \dots, 0\}$ , that there exists an integer  $\mu \in [0, 2^r L]$  such that

$$f(x) = p(x), \quad x \in [\alpha, \beta],$$

where

$$\alpha = \max\{0, 2^{R-r}(\mu - 2n + 2)\}, \quad \beta = \min\{2^r L, 2^{R-r}(\mu + 2n - 2)\}.$$

Then, using also (7.34), (7.30), (6.3) and (6.5), we find

$$\begin{aligned} d_{\mu}^{(r)} &= p(2^{R-r-1}(2\mu+1)) - \sum_{k=\max\{0, \mu-2n+2\}}^{k=\min\{2^r L, \mu+2n-2\}} p(2^{R-r}k) \phi_k^r(\mu + \tfrac{1}{2}) \\ &= p(2^{R-r-1}(2\mu+1)) - \sum_{k \in \mathbb{Z}_{2^r L}} p(2^{R-r}k) \phi_k^r(\mu + \tfrac{1}{2}) = 0. \end{aligned}$$

Hence the interpolation wavelet coefficients  $d_j^{(r)}$  of the function  $f$ , with local polynomial-like behaviour, can be expected to be small in comparison to the interpolation wavelet coefficients  $d_j^{(r)}$  corresponding to those regions where the function  $f$  exhibits non-polynomial-like behaviour.

For example, choose the function  $f$  in (7.34) as the cubic cardinal B-spline  $N_4$ , with an arbitrary choice of the integers  $R$  and  $L$ . As follows from (1.14) and (1.10), the B-spline  $N_4$  consists of four cubic polynomial pieces joined together in such a way that the second derivative of the B-spline  $N_4$  is continuous, while the third derivative has jump discontinuities at the integers  $\mathbb{Z}_4$ . Now choose  $n = 2$  to ensure, by the above argument, that the wavelet coefficients  $d_j^{(r)}$ ,  $r = R - 1, R - 2, \dots, 0$  are zero in the regions where  $f$  is identical to a polynomial and non-zero in the regions where the third derivative has a jump discontinuity.

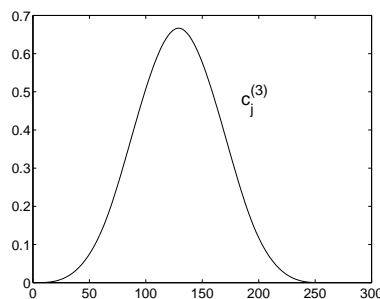


Figure 7.2: Cubic cardinal B-spline  $N_4$ , sampled at 257 equally spaced points.

In Figure 7.2, we have chosen  $R = 3$  and  $L = 32$  and plotted the sequence of sampled values  $c_j^{(3)} = f(j)$  versus their indices. Figure 7.3 shows the result of the interpolation wavelet decomposition algorithm (7.34), where the resulting coefficients are also plotted versus their indices.

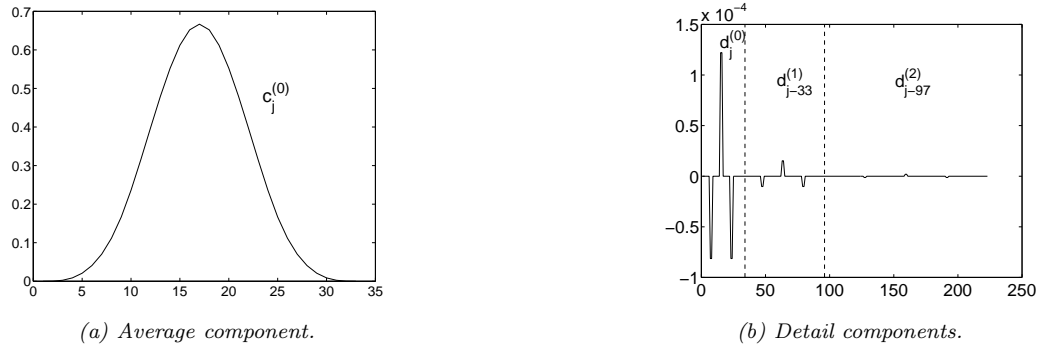


Figure 7.3: Third level decomposition of the cubic cardinal B-spline  $N_4$  of Figure 7.2.

We observe that the average values  $c_j^{(0)}$ , as shown in Figure 7.3(a), consist of precisely every eighth value of the original B-spline. What is of more interest is the set of detail components shown in Figure 7.3(b). Note that, as expected, the interpolation wavelet coefficients  $d_j^{(r)}$ ,  $r = 2, 1, 0$ , are non-zero only where the B-spline  $N_4$  has jump discontinuities in its third derivative.

## 7.4 Examples

In this section we illustrate the interpolation wavelets by applying them to two practical problems. Of course we do not claim that the schemes described above are more efficient in practice than any of the alternatives. This would require a detailed study which we leave for future consideration. These examples merely serve to illustrate the theory developed above. Note in particular how the interpolation wavelet decomposition on an interval avoids any edge artifacts.

### 7.4.1 Signature smoothing

Figure 7.4(a) shows part of a signature that was captured by a digitized tablet. One clearly sees the quantization effect of the underlying grid of this particular tablet. Almost all applications require that the signature should be smoothed. We therefore applied a single level of the interpolation wavelet decomposition (7.34) with  $n = 2$ . All the detail

coefficients were set equal to zero, and a single reconstruction step was performed using the zero detail coefficients. The result is the smoothed signature in Figure 7.4(b). Note that we follow standard practice by displaying only the discrete coefficients  $c_j^{(R)}$ , connected by straight line segments. This is different from the smooth curve given by (7.36).

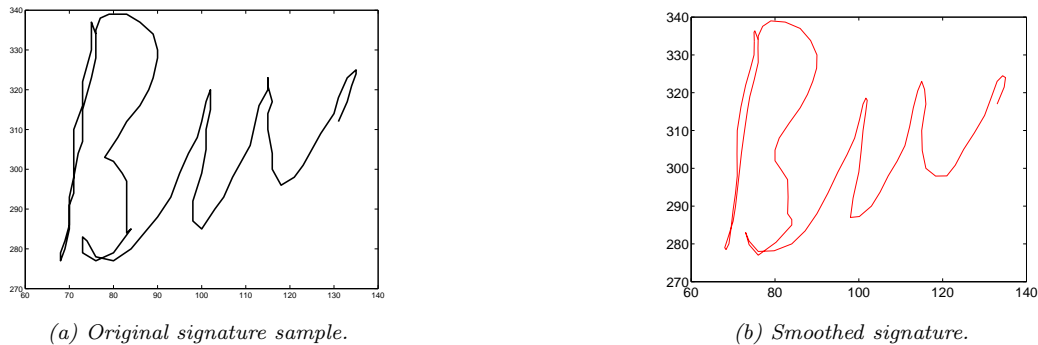
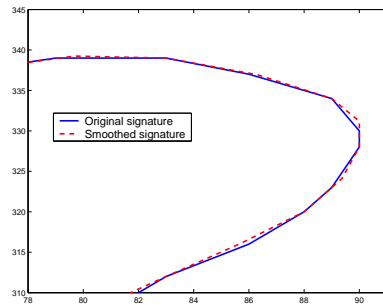


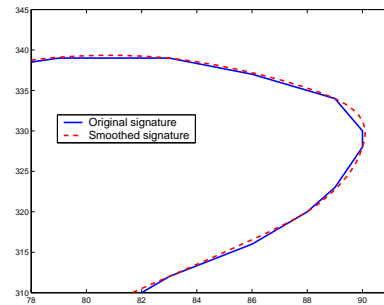
Figure 7.4: Illustration of smoothing with interpolation wavelet decomposition.

A closer look at the original and reconstructed signature is given in Figure 7.5. Note from Figure 7.5(a) how every other data point stays the same (due to the interpolation property). The remaining points however, are calculated in such a way that the result is a smoother signature. In fact, this procedure is exactly the same as if we discarded every other data point of the original signature and then applied one step of the subdivision scheme with  $n = 2$  to the result, as described in Chapter 6. As mentioned above we plot only the data points connected by straight lines, and not the smooth curve  $f_R$  described by (7.36). An efficient way of calculating  $f_R$  is by subdivision: according to the argument leading from (7.36) to (7.38), each step of the subdivision scheme doubles the number of points on the curve  $f_R$ . The result of two more subdivision steps is shown in Figure 7.5(b). Note how the reconstructed curve has become noticeably smoother.

Note that the main problem in this case is that the data points themselves are corrupted by noise. Insisting that the data points are interpolated is therefore not the best way to proceed. In this case non-interpolatory schemes such as Lane–Riesenfeld [45] (see also [50, Chapter 2]) might prove beneficial. Even in this less than ideal situation, the interpolation wavelets provide a surprisingly good smoothing, with no artificial edge effects.



(a) A single reconstruction step



(b) After two more subdivision steps.

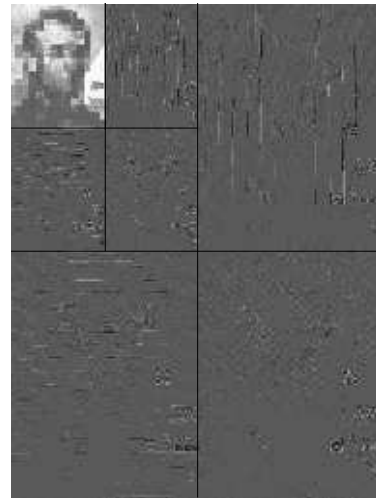
Figure 7.5: Magnification of original and smoothed signature.

## 7.4.2 Two-dimensional interpolation wavelet decomposition

For our second example we use the well-known painting by Salvador Dali, *Gala contemplating the Mediterranean*, Figure 7.6(a), the original of which can be seen in the Dali museum in Figueras, Spain. Although already painted in 1976, it is a beautiful illustration of Dali's awareness of images on different scales, in this case, a portrait of Abraham Lincoln, the 16th President of the United States of America, and Gala, Dali's wife, looking out to sea.



(a) Original image.



(b) Decomposition of image.

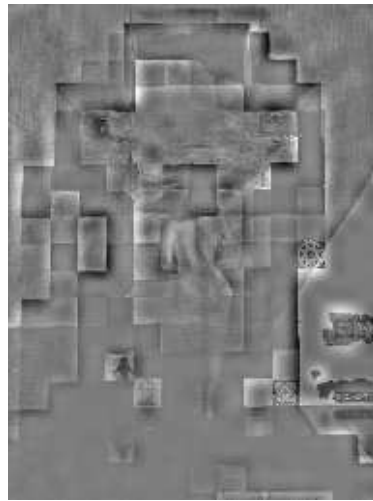
Figure 7.6: Illustration of two-dimensional interpolation wavelet decomposition.

Constructing a two-dimensional tensor product from the interpolation wavelet (7.10) (see e.g. [9, Section 6.4]), allows us to perform a two-dimensional decomposition of Dali's

painting, as illustrated in Figure 7.6(b), effectively separating the high frequency components (detail) and the low frequency components (average). (With the original painting, the same effect is obtained by viewing it from a distance, again effectively removing the detail components—obviously what Dali had in mind.) Note how clearly the image of Abraham Lincoln is captured by the average component, displayed in the top left-hand corner of Figure 7.6(b). The remaining part of the figure consists of the various detail components.



(a) *Detail removed.*



(b) *Average removed.*

Figure 7.7: Reconstructed images.

Now we set the detail components equal to zero, and then reconstruct according to a two-dimensional reconstruction algorithm also based on the tensor product interpolation wavelet. The result, as shown in Figure 7.7(a), is the image of Abraham Lincoln interpolated back onto the original grid of Figure 7.6(a). If we set the average component equal to zero, and then reconstruct, we obtain the image in Figure 7.7(b), which contains all the areas of sharp transition which are absent from Figure 7.7(a).



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